

FRACTIONAL HELLY PROPERTY AND COMBINATORICS OF FORKING IN NTP_2 THEORIES

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ABSTRACT. We investigate the class of FHP theories, i.e. theories of structures in which all definable families of sets satisfy the Fractional Helly Property (and its variants) from combinatorics. FHP theories generalize NIP and form a new subclass of low NTP_2 theories. We give many new examples (including ultraproducts of finite fields and of the p -adics) and establish some results about forking and f -generics for amenable groups definable in FHP theories. We make several conjectures about finitary combinatorial properties of forking in NTP_2 theories and establish some partial results, as well as investigate related two-cardinal type counting functions addressing a question of Adler.

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1. INTRODUCTION

The *fractional Helly theorem* is a basic compactness principle in discrete geometry: if a positive proportion of the small subfamilies of a finite family of sets have non-empty intersection, then one can find a large intersecting subfamily (see Section 2). It was initially established for convex subsets of \mathbb{R}^d by [KL79] (with optimal bounds established in [Kal84, Eck85]), and since then extended to many other families of sets of geometric or combinatorial interest. We refer to excellent surveys [ADLS17, DLGMM19, BK22] for a detailed discussion and references.

In particular, for families of finite VC-dimension, Matoušek [Mat04] showed that an analogous conclusion still holds, and this viewpoint has played an important role in the interaction between combinatorial geometry, VC-theory, and model theory of NIP structures (in particular through its application to the so called “ (p, q) -theorem” for families of finite VC dimension - see Section 4.2). The aim of this paper is to study the model-theoretic content of the fractional Helly phenomenon beyond the NIP context and to use it as a tool in the study of forking in NTP_2 theories.

Namely, we say that a partitioned formula $\varphi(x, y) \in L$ has the fractional Helly property (FHP) in a structure M if the definable family $\{\varphi(M, b) : b \in M^y\}$ of subsets of M^x defined by its instances satisfies a fractional Helly theorem; and a complete theory $T = \text{Th}(M)$ is FHP if every partitioned formula is. In Section 2 we consider basic properties of FHP formulas and theories, along with some variants of the property). We place FHP theories inside Shelah’s classification hierarchy: FHP theories form a proper subclass of low NTP_2 theories containing NIP theories (Propositions 2.21 and 2.25); and show that FHP for a theory reduces to checking that all formulas $\varphi(x, y)$ with x singleton are FHP (Lemma 2.11).

In Section 3 we formulate a relative version of the FHP property with respect to a class of measures (where the original property corresponds to the class of finitely supported measures, Proposition 3.6), and connect it with the theory of generically stable Keisler measures in NIP theories. In particular, in Section 3.3 we demonstrate that Matoušek’s theorem (FHP for families of finite VC dimension) implies a uniform/local version of the main result about generically stable measures from Hrushovski-Pillay-Simon [HPS12]; and conversely its qualitative version under the global NIP assumption follows from [HPS12] by compactness.

In Section 4 we establish a connection between the so-called “colorful” fractional Helly property from combinatorics and the notion of burden/dp-rank from the study of NIP/ NTP_2 theories. In particular, generalizing and refining a measure theoretic characterization of strong NIP theories by Pillay [Pil13], we get (which can also be viewed as a multi-measure generalization of [HPS12]):

Theorem (Proposition 4.6). *Assume that $\text{bdn}(\mathbb{M}^x) < k$ and let $\varphi_i(x, y_i) \in L$, $i \in [k]$, satisfy FHP relatively to a class of definable measures $\mathfrak{M}_i \subseteq \mathfrak{M}_{y_i}(\mathbb{M})$ (see Definition 3.5). Then for every $\alpha > 0$ there is $\gamma > 0$ satisfying the following. Let $\mu_i \in \mathfrak{M}_i$ be such that μ_1, \dots, μ_k are pairwise commuting (in particular each μ_i commutes with itself). Assume that $\mu_1 \otimes \dots \otimes \mu_k \left(\exists x \bigwedge_{i=1}^k \varphi_i(x, y_i) \right) \geq \alpha$. Then there is some $i \in [k]$ and some $a \in \mathbb{M}^x$ such that $\mu_i(\varphi_i(a, y_i)) \geq \gamma$.*

In particular, this implies that families of finite VC-dimension satisfy colorful fractional Helly property (Corollary 4.12) and gives a bound on the fractional Helly number of formulas in terms of the burden/dp-rank (rather than the dual VC-density, as in Matoušek's theorem; see the discussion after Remark 4.8):

Theorem (Corollary 4.10). *In any FHP (so e.g. in NIP) theory T , the fractional Helly number of a formula $\varphi(x, y)$ is at most $\text{bdn}(\mathbb{M}^x) + 1$.*

In Section 4.2 we consider one of the main applications of the fractional Helly property, Matousek's (p, q) -theorem for families of finite VC-dimension (Fact 4.14, which is an analog of the Alon and Kleitman's (p, q) -theorem for convex sets in \mathbb{R}^d [AK92]). Matousek's result plays an important role in the study of NIP theories (see the references there). As we demonstrate in this paper, the class of FHP structures is much wider than the class of NIP structures. However, at the level of the theory, the (p, q) -theorem characterizes NIP/finite VC dimension:

Theorem (Proposition 4.16). *Assume that the formula $\psi(x; y_1, y_2) := \varphi(x, y_1) \wedge \neg\varphi(x, y_2)$ satisfies the (p, q) -theorem. Then $\varphi(x, y)$ is NIP (i.e. its instances define a family of sets of finite VC-dimension).*

In Section 5 turn to definable groups and forking. Several notions of largeness/genericity for definable sets (and their equivariant versions in definable groups), coming from combinatorics, topological dynamics or measure theory, play an important role in the model-theoretic study of tame classes of structures. In definably amenable NIP groups all of these notions agree, giving a canonical notion of a large set ([CS18], see Section 5.1 for the details). Here we extend the connection of forking and invariant measures to amenable FHP groups:

Theorem (Theorem 5.11). *Assume T is FHP and $G = G(\mathbb{M})$ is a definable group so that $G(M)$ is amenable (as a discrete group) for some $M \models T$. Then for any $L(\mathbb{M})$ -definable set $X \subseteq G(\mathbb{M})$, X is f -generic if and only if $\mu(X) > 0$ for some G -invariant measure μ .*

In Section 5.4 we note a partial analog for the action of $\text{Aut}(\mathbb{M})$.

The so called fsg groups (groups with finitely satisfiable generics), arising in the work on Pillay's o -minimal groups conjecture [HPP08], form a particularly nice class of definably amenable NIP groups capturing definable compactness in many natural settings (including o -minimal theories). In [CGK24], a generalization of fsg groups from NIP theories to *fm groups* (see Section 5.3) in arbitrary theories was proposed, demonstrating that part of the theory of fsg groups in NIP theories survives. Here we show that in FHP theories, the characterization of generic definable sets generalizes fully from the NIP case:

Theorem (Proposition 5.25). *Let G be a definable fm group in an FHP theory. Then all notions of genericity (1)–(5) in Definition 5.5 are equivalent for definable subsets of G .*

In Section 6 we consider the fractional Helly property for definable sets in two (simple, unstable) expansions of $(\mathbb{Z}, +)$: by a predicate Pr for the primes and their inverses (studied in [KS17], see Section 6.2), and by a predicate Sqf for the square-free integers (studied in [BT21], see Section 6.1). They exhibit two quite different behaviors, explained by the former set being of Banach density 0, while the latter is of positive Banach density:

Theorem (Theorem 6.8). *The structure $(\mathbb{Z}, +, \text{Sqf})$ is FHP. Moreover, every formula $\varphi(x, y)$ with $|x| \leq d$ satisfies FHP_{d+1} .*

Theorem. *The structure $(\mathbb{Z}, +, \text{Pr})$ is not FHP (Proposition 6.11), and (assuming Dickson's conjecture) T_{Pr} is locally FHP (Theorem 6.16).*

In Section 7 we consider FHP in measurable structures in the sense of Macpherson and Steinhorn, or *MS-measurable structures* [MS08]. Main examples of MS-measurable structures are ultraproducts of finite fields, finite simple groups of bounded Lie rank, vector spaces, etc. (we refer to [EM08, Example 2.4] for further examples).

Theorem (Theorem 7.6). *Let M be an MS-measurable structure. Then every partitioned formula $\varphi(x, y) \in L$ with $|x| \leq d$ satisfies FHP_{d+1} with respect to the class of definable measures $\mathfrak{M}_y := \{\mu_B(y) : B \subseteq \mathbb{M}^y \text{ definable with parameters}\}$ (see Definition 3.5). In particular, $\varphi(x, y)$ satisfies FHP_{d+1} .*

As an application, we get that definable families of sets of bounded description complexity in large finite fields satisfy the fractional Helly property:

Theorem (see Corollary 7.9 for the precise statement). *For every $D \in \mathbb{N}$ and $\alpha > 0$ there exist $\beta = \beta(D, \alpha) > 0$ so that: if F is a sufficiently large finite field and $\mathcal{F} \subseteq F^d$ a definable family of sets of description complexity $\leq D$ so that $|\{I \subseteq \mathcal{F} : |I| = d+1 \wedge \bigcap_{S \in I} S \neq \emptyset\}| \geq \alpha \binom{|\mathcal{F}|}{d+1}$, then $\bigcap_{S \in J} S \neq \emptyset$ for some $J \subseteq \mathcal{F}$ with $|J| \geq \beta |\mathcal{F}|$.*

In Section 8 we consider the fractional Helly property for definable families of sets in valued fields (we note that for special definable families, namely convex sets in the sense of Monna, fractional Helly property was studied in [CM23]). We prove an Ax-Kochen-Ershov style result for the FHP property in henselian valued fields:

Theorem. (Theorem 8.5) *Let K be an equi-characteristic 0 henselian valued field. Then K satisfies FHP if and only if both the residue field k and the (ordered) value group Γ satisfy FHP.*

Combining this with the aforementioned result for FHP in pseudo-finite fields, the existing burden calculations [Che10, Che14, CH14, CS19] and Corollary 4.10, we obtain the following explicit bounds for the ultraproducts of the p -adics (which are NTP_2 but not NIP, so Matoušek's theorem does not apply):

Theorem. (Corollary 8.13) *Let K be $\prod_{i \in \mathbb{N}} \mathbb{Q}_{p_i} / \mathcal{U}$ or $\prod_{i \in \mathbb{N}} \mathbb{F}_{p_i}((t)) / \mathcal{U}$ for some prime p_i and a non-principal ultrafilter \mathcal{U} on \mathbb{N} . Then every partitioned formula $\varphi(x, y) \in L$ in K with $|x| \leq d$ satisfies FHP_{2d} .*

We expect the optimal bound to be FHP_{d+1} here (Conjecture 8.14).

The *UDTFS* property, or *Uniform Definability of Types over Finite Sets* (conjectured by Laskowski, and established in [CS13, CS15]), suggests that in NIP

theories local types over finite sets behave similarly to stable theories (see Section 9.2). It plays an important role in the study of NIP theories and provides a model theoretic counterpart for the existence of compression schemes for families of finite VC-dimension in theoretical machine learning. In this section we propose a generalization of this conjecture for NTP_2 theories, aiming to capture that, when restricted to finite sets in a uniform manner, dividing in NTP_2 theories behaves similarly to dividing in simple theories. We call this property the *Uniform Local Character (of dividing) over Finite Sets*, or *ULCFS* (Section 9). We show that both simple (Proposition 9.9) and NIP theories (Proposition 9.7) satisfy (strong) ULCFS, and that ULCFS implies NTP_2 (Proposition 9.15). We also discuss resilience in Section 9.4, showing that it reduces to formulas in one variable (Corollary 9.14) and is implied by strong ULCFS (Proposition 9.15). We conjecture that all NTP_2 theories satisfy ULCFS (Conjecture 9.17). In Section 9.5, inspired by the classical infinitary two-cardinal function considered in Section 10, we consider a two-parameter function $f_\varphi(k, l)$ with $k \leq l \in \mathbb{N}$ counting the number of pairwise-inconsistent partial φ -types of size k over a set of parameters of size l (which in the case $k = l$ corresponds to the dual shatter function π_φ^*). In the same way as UDTFS is a strengthening of the Sauer-Shelah lemma, we view ULCFS as a strengthening of $f_\varphi(k, l)$ being bounded by a polynomial $p(l)$ of degree independent of k , for $l \gg k$ (Definition 9.18). We observe that indeed its polynomial boundedness implies NTP_2 (Proposition 9.21) and follows from ULCFS (Proposition 9.23). Again, we conjecture that NTP_2 implies polynomial boundedness of $f_\varphi(k, l)$ (Conjecture 9.25) and obtain a partial result that at least NTP_2 implies ε -power saving (so NTP_2 is characterized by the non-maximality of $f_\varphi(k, l)$, Proposition 9.28).

In Section 10 we consider a related two-cardinal partial type counting function: for a theory T and infinite cardinals $\kappa \leq \lambda$, we let $f_T(\kappa, \lambda)$ be the supremum of the cardinalities $|P|$, where P is a family of pairwise inconsistent partial types each of cardinality $\leq \kappa$, all over the same fixed set of parameters of size λ . A celebrated result of Keisler [Kei76], refining earlier work of Shelah [She71] and Morley [Mor65], demonstrates that restricting to the case $\kappa = \lambda$, there are exactly six possibilities for f_T when T is a complete countable theory. Shelah also proved that simplicity of the theory T is detected by $f_T(\kappa, \lambda)$. Following this, Adler [Adl09] conjectured (see Conjecture 10.6) that NTP_2 can be detected by f_T , and that there are only finitely many possibilities for f_T when T is countable. In Proposition 10.15 we significantly narrow down the possibilities for $f_T(\kappa, \lambda)$ (and conjecture that this is a complete list of possibilities). In particular, our result refutes the former conjecture of Adler (Corollary 10.18), and confirms the latter one under the GCH assumption (Corollary 10.17).

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2. BASIC PROPERTIES OF FHP FORMULAS AND RELATION TO SHELAH'S CLASSIFICATION

Our notation is mostly standard. Given $n \in \mathbb{N}$, we write $[n]$ to denote the set $\{1, 2, \dots, n\}$. We let x, y, z, \dots denote finite tuples of variables, and if M is a first-order structure then M^x denotes the corresponding sort of M . For $a, b \in M^x$ and $A \subseteq M$, we write $a \equiv_A b$ if $\text{tp}(a/A) = \text{tp}(b/A)$.

2.1. Basic properties of FHP formulas. Given a set X and $k \in \mathbb{N}$, we let $\binom{X}{k}$ denote the set of all k -element subsets of X . Given a tuple $\bar{S} = (S_i : i \in I)$ of subsets of X (possibly with repetitions) and $k \in \mathbb{N}$, let $\text{Cons}_k(\bar{S}) := \left\{ J \in \binom{I}{k} : \bigcap_{i \in J} S_i \neq \emptyset \right\}$. We recall the fractional Helly property from combinatorics (see e.g. [AKMM02]):

Definition 2.1. Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a (possibly infinite) family of subsets of X .

- (1) For $k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}_{>0}$, the family \mathcal{F} satisfies $\text{FHP}(k, \alpha, \beta)$ (where FHP stands for *Fractional Helly Property*) if the following holds: for any $n \in \mathbb{N}$ and any tuple $\bar{S} = (S_1, \dots, S_n)$ of sets in \mathcal{F} (possibly with repetitions), if $|\text{Cons}_k(\bar{S})| \geq \alpha \binom{n}{k}$, then there is some $J \subseteq [n]$ with $|J| \geq \beta n$ such that $\bigcap_{i \in J} S_i \neq \emptyset$.
- (2) For $k \in \mathbb{N}$, we say that \mathcal{F} satisfies FHP_k if for every $\alpha \in \mathbb{R}_{>0}$ there is some $\beta \in \mathbb{R}_{>0}$ so that \mathcal{F} satisfies $\text{FHP}(k, \alpha, \beta)$.
- (3) We say that \mathcal{F} satisfies FHP if it satisfies FHP_k for some $k \in \mathbb{N}$. The smallest k so that \mathcal{F} satisfies FHP_k is the *fractional Helly number* of \mathcal{F} .

We will also consider the following weakening:

- Definition 2.2.**
- (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfies the (p, k) -property, where $p \geq k \in \mathbb{N}$, if for any tuple $\bar{S} = (S_1, \dots, S_p)$ of sets in \mathcal{F} there is some $I \subseteq [p]$, $|I| = k$ so that $\bigcap_{i \in I} S_i \neq \emptyset$.
 - (2) For $k, p \in \mathbb{N}$ and $\beta \in \mathbb{R}_{>0}$, the family \mathcal{F} satisfies $\text{WFHP}(k, p, \beta)$ (*Weak Fractional Helly Property*) if the following holds: for any $n \in \mathbb{N}$ and any tuple $\bar{S} = (S_1, \dots, S_n)$ of sets in \mathcal{F} , if $\{S_1, \dots, S_n\}$ satisfies the (p, k) -property, then there is some $J \subseteq [n]$ with $|J| \geq \beta n$ such that $\bigcap_{i \in J} S_i \neq \emptyset$.
 - (3) For $k \in \mathbb{N}$, we say that \mathcal{F} satisfies WFHP_k if there exist some $p_0 \geq k \in \mathbb{N}$ so that: for every $p \geq p_0$ there is $\beta \in \mathbb{R}_{>0}$ so that \mathcal{F} satisfies $\text{WFHP}(k, p, \beta)$; and \mathcal{F} satisfies WFHP if it satisfies WFHP_k for some $k \in \mathbb{N}$.

The following two remarks are (rephrased) from e.g. [AK92, AKMM02]:

Proposition 2.3. *If \mathcal{F} satisfies FHP_k , then it also satisfies WFHP_k .*

Proof. Assume \mathcal{F} satisfies the (p, k) -property for some p , and let $\bar{S} = (S_1, \dots, S_n)$ be any finite tuple of sets from \mathcal{F} . As every p -tuple of sets from \bar{S} contains at least one k -tuple with a non-empty intersection, and each k -tuple is contained in $\binom{n-k}{p-k}$ of the p -tuples, by double counting there are at least $\frac{\binom{n}{p}}{\binom{n-k}{p-k}}$ k -tuples with non-empty intersections. Taking $p_0 = p_0(k) \in \mathbb{N}$ sufficiently large, and assuming that $p \geq p_0$ and $\alpha = \alpha(p, k) > 0$ sufficiently small, this gives at least $\alpha \binom{n}{k}$ k -tuples with non-empty intersections. As \mathcal{F} satisfies FHP_k , there is some $\beta = \beta(\alpha) = \beta(p, k) > 0$ and $J \subseteq [n]$ with $|J| \geq \beta n$ and $\bigcap_{i \in J} S_i \neq \emptyset$. \square

Remark 2.4. We could have defined the (p, k) -property requiring instead that the sets appearing in \bar{S} in Definition 2.2 are pairwise distinct — this would not change the definition of WFHP _{k} : if \mathcal{F} satisfies the (p, k) -property without repetitions, then for $p' := k(p - 1) + 1$ it satisfies the (p', k) -property with repetitions. Indeed, let $\bar{S} = (S_1, \dots, S_n)$ be any finite tuple of sets from \mathcal{F} , possibly with repetitions. Let $p' := k(p - 1) + 1$. Then for any $I \subseteq [n], |I| = p'$ there is some $J \subseteq I$ with $|J| = k$ and $\bigcap_{i \in J} S_i \neq \emptyset$. Indeed, either J contains k copies of the same (non-empty) set which clearly intersect, or p distinct sets, in which case some k among them have a non-empty intersection by assumption.

We also note that the assumption on the family in the WFHP property is strictly stronger than the assumption in the FHP property:

Lemma 2.5. *Fix integers $k \geq 2$ and $p' \geq k \geq 2$, and reals $\alpha \in (0, 1)$ and $\gamma \in (0, 1]$. Then there exist arbitrarily large $n \in \mathbb{N}$ and $\{S_1, \dots, S_n\} \subseteq \mathcal{P}(X)$ such that:*

(1) *More than an α -fraction of the k -subfamilies intersect, i.e.*

$$|\{I \subseteq [n] : |I| = k \text{ and } \bigcap_{i \in I} S_i \neq \emptyset\}| > \alpha \binom{n}{k}.$$

(2) *For every $J \subseteq [n]$ with $|J| \geq \gamma n$, the induced subfamily $\{S_i : i \in J\}$ fails the (p', k') -property.*

Proof. Choose a sufficiently large integer $r \geq k$ so that

$$(2.1) \quad \prod_{j=0}^{k-1} \left(1 - \frac{j}{r}\right) > \alpha.$$

Next choose any integer $m \geq \lceil p'/\gamma \rceil$ and set $n := rm$. Partition $[n]$ into r disjoint “blocks” B_1, \dots, B_r , each of size m . Let

$$X := \{E \subseteq [n] : |E| = k \text{ and } |E \cap B_t| \leq 1 \text{ for every } t \in \{1, \dots, r\}\},$$

and for each $i \in [n]$ define $S_i := \{E \in X : i \in E\}$.

For any $I \subseteq [n]$ with $|I| = k$ we claim

$$(2.2) \quad \bigcap_{i \in I} S_i \neq \emptyset \iff I \text{ meets } k \text{ distinct blocks.}$$

Indeed, if I has one element in each of k distinct blocks, then $I \in X$ and $I \in S_i$ for all $i \in I$, so the intersection contains I . Conversely, if I contains two elements from the same block B_t , then no $E \in X$ can contain both, hence $\bigcap_{i \in I} S_i = \emptyset$.

By (2.2), the intersecting k -tuples are obtained by choosing k blocks and then one element from each chosen block. Thus

$$\begin{aligned} |\{I \subseteq [n] : |I| = k, \bigcap_{i \in I} S_i \neq \emptyset\}| &= \binom{r}{k} m^k, \text{ and using } n = rm, \frac{\binom{r}{k} m^k}{\binom{n}{k}} = \\ &= \frac{r(r-1) \cdots (r-k+1) m^k}{rm(rm-1) \cdots (rm-k+1)} > \frac{r(r-1) \cdots (r-k+1) m^k}{(rm)^k} = \prod_{j=0}^{k-1} \left(1 - \frac{j}{r}\right). \end{aligned}$$

Together with (2.1) this gives (1).

Let now $J \subseteq [n]$ satisfy $|J| \geq \gamma n = \gamma rm$. Then some block B_t contains at least $|J \cap B_t| \geq \frac{|J|}{r} \geq \gamma m \geq p'$ elements of J . Choose distinct indices $i_1, \dots, i_{p'} \in J \cap B_t$. If $a \neq b \in [p']$, then $S_{i_a} \cap S_{i_b} = \emptyset$, since any $E \in X$ contains

at most one element from the block B_i . In particular, for every $L \subseteq \{1, \dots, p'\}$ with $|L| = k' \geq 2$ we have $\bigcap_{\ell \in L} S_{i_\ell} = \emptyset$. This witnesses that the induced subfamily $\{S_i : i \in J\}$ fails the (p', k') -property, proving (2). \square

We now specialize these notions to definable families of sets:

- Definition 2.6.** (1) Let $\mathcal{M} = (M, \dots)$ be a first-order structure in a language \mathcal{L} . We say that a partitioned formula $\varphi(x, y) \in \mathcal{L}$ (with x, y arbitrary finite tuples of variables) satisfies FHP(k, α, β) (FHP _{k} , FHP, etc.) in \mathcal{M} if the corresponding family of definable sets $\mathcal{F}_\varphi := \{\varphi(M, b) : b \in M^y\} \subseteq \mathcal{P}(M^x)$ does (where $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$).
- (2) An \mathcal{L} -theory T is FHP (WFHP) if every partitioned formula $\varphi(x, y)$ with x, y arbitrary finite tuples of variables satisfies FHP (respectively, WFHP) in every model of T .

Remark 2.7. Note that if $\varphi(x, y)$ satisfies FHP(k, α, β) in \mathcal{M} and $\mathcal{N} \equiv \mathcal{M}$, then $\varphi(x, y)$ satisfies FHP(k, α, β) in \mathcal{N} , hence for a complete theory it suffices to verify FHP in a single model (and similarly for WFHP).

Recall that, for $A \subseteq \mathbb{M} \models T$ and $b \in \mathbb{M}^y$, a formula $\varphi(x, b)$ *k-divides over A* if there is an infinite sequence $(b_i : i \in \mathbb{N})$ in \mathbb{M}^y with $b_i \equiv_A b$ so that the set of formulas $\{\varphi(x, b_i) : i \in \mathbb{N}\}$ is *k-inconsistent*; and $\varphi(x, b)$ *divides over A* if it *k-divides over A* for some k . We refer to e.g. [CK12] for the basic properties of dividing. The following was observed in the proof of [CS15, Proposition 25], and is straightforward by Ramsey and compactness:

Remark 2.8. Given a formula $\varphi(x, y)$, $k \in \mathbb{N}$ and a partial type $\pi(y)$ over a small set of parameters A , the family $\mathcal{F}_{\varphi, \pi} := \{\varphi(\mathbb{M}, b) : b \models \pi(y)\}$ satisfies the (p, k) -property for some p if and only if for every $b \models \pi$, $\varphi(x, b)$ does not *k-divide*.

Definition 2.9. Let T be a complete theory, $\mathbb{M} \models T$ a monster model and $A \subseteq \mathbb{M}$ a small set of parameters.

- (1) A partitioned formula $\varphi(x, y)$ satisfies *local FHP _{k} over A* (respectively, *local WFHP _{k} over A*) if for every complete type $q(y) \in S_y(A)$, the corresponding family of definable sets $\mathcal{F}_{\varphi, q} := \{\varphi(\mathbb{M}, b) : b \in \mathbb{M}^y, b \models q\}$ satisfies FHP _{k} (respectively, WFHP _{k}). A theory T is *locally FHP* (*locally WFHP*) if every formula satisfies local FHP _{k} (local WFHP _{k}) over \emptyset for some $k = k(\varphi) \in \mathbb{N}$ (then FHP holds over all sets of parameters).
- (2) A partitioned formula $\varphi(x, y)$ satisfies FHP _{k} *for dividing*, or DFHP _{k} , over A if the following holds: for any $q \in S_y(A)$ so that $\varphi(x, b)$ does not *k-divide* over A for some/any $b \models q$, there is some $\beta = \beta(q) \in \mathbb{R}_{>0}$ so that for any finite tuple B of realizations on q there is some $B_0 \subseteq B, |B_0| \geq \beta|B|$ with $\{\varphi(x, b) : b \in B_0\}$ consistent. And $\varphi(x, y)$ satisfies DFHP over A if it satisfies DFHP _{k} over A for some k .

Remark 2.10. (1) Immediately from the definitions we have: if $\varphi(x, y)$ satisfies FHP (WFHP), then it also satisfies local FHP (respectively, local WFHP) over all sets of parameters. The converse is true if T is ω -categorical (see Proposition 2.31), but not in general (see Example 2.30).

- (2) If $\varphi(x, y)$ satisfies local FHP over A , then it also satisfies local WFHP over A (as in Proposition 2.3).

- (3) If $\varphi(x, y)$ satisfies local WFHP_{*k*} over A , then it also satisfies DFHP_{*k*} over A . Indeed, assume $\varphi(x, y)$ satisfies local WFHP_{*k*} over A . If $\varphi(x, b)$ does not k -divide over A , then by Remark 2.8 the family $\mathcal{F}_{\varphi, \text{tp}(b/A)}$ satisfies the (p, k) -property for some $p \geq k$, without loss of generality $p \geq p_0$. As by assumption φ satisfies WFHP(k, p, β) over A for some $\beta > 0$, we conclude.
- (4) If T is FHP (WFHP), then any reduct of T is also FHP (respectively, WFHP).

- Lemma 2.11.** (1) *If $k \leq k'$ and $\varphi(x, y)$ satisfies (local) FHP_{*k*}, then it satisfies (local) FHP_{*k'*} as well. And the same for (local) WFHP_{*k*} and DFHP_{*k*}.*
- (2) *Both classes of formulas with FHP and with local FHP are closed under disjunctions. More precisely, if $\varphi_i(x, y_i)$ satisfies (local) FHP_{*k_i*}, for $1 \leq i \leq t$, then $\varphi(x; y_1, \dots, y_t) := \bigvee_{i=1}^t \varphi_i(x, y_i)$ satisfies (local) FHP_{*k*} for $k := \sum_{1 \leq i \leq t} k_i - t + 1$.*
- (3) *Given a complete theory T , if every formula $\varphi(x, y)$ with $|x| = 1$ singleton satisfies FHP, then every formula satisfies FHP (and if every formula $\varphi(x, y)$ with $|x| = 1$ satisfies FHP_{*k*}, then every formula $\psi(x', y)$ with $|x'| = n$ satisfies FHP_{*kⁿ*}).*
- (4) *If T satisfies FHP, then any reduct of T and T^{eq} satisfies FHP.*

Proof. (1) Let $\varphi(x, y)$ and $k < k'$ be given. Fix an arbitrary $\alpha > 0$. Let $B \subseteq \mathbb{M}^y$ be a finite set, $n = |B|$, $\bar{S} := (\varphi(M, b) : b \in B)$ such that $\text{Cons}_{k'}(\bar{S}) \geq \alpha \binom{n}{k'}$. Obviously every $J \in \text{Cons}_{k'}(\bar{S})$ contains some $J' \subseteq J$ with $J' \in \text{Cons}_k(\bar{S})$, and for every $J' \in \binom{B}{k}$ there are at most $\binom{n-k}{k'-k}$ sets $J \in \binom{B}{k'}$ that contain it. Hence, by double counting and basic properties of the binomial coefficients we have that

$$|\text{Cons}_k(\bar{S})| \geq \frac{\alpha \binom{n}{k'}}{\binom{n-k}{k'-k}} \geq \alpha' \binom{n}{k}$$

for some $\alpha' = \alpha'(\alpha, k, k') > 0$ and all $n \in \mathbb{N}$. Then taking $\beta > 0$ so that $\varphi(x, y)$ satisfies FHP(k, α', β), we get that $\varphi(x, y)$ also satisfies FHP(k', α, β). The proof for local FHP is identical, restricting to B a tuple of realizations of a complete type $q(y)$. For (local) WFHP_{*k*}, we only need to note that any family satisfying the (p, k') -property also satisfies the (p, k) -property. For DFHP, note that if $\varphi(x, b)$ does not k' -divide over A , then it also does not k -divide over A .

(2) Let $\varphi(x, \bar{y})$, $\bar{y} = (y_1, \dots, y_t)$ and k be as in the statement, and fix some $\alpha > 0$. Let B be a finite set of tuples $\bar{b} = (b_1, \dots, b_t)$, $n := |B|$, and let $\bar{S} := (\varphi(\mathbb{M}, \bar{b}) : \bar{b} \in B)$ be such that $\text{Cons}_k(\bar{S}) \geq \alpha \binom{n}{k}$. For $1 \leq i \leq t$, let $\bar{S}_i := (\varphi_i(\mathbb{M}, b_i) : \bar{b} \in B)$. By the choice of k , for each $J \in \text{Cons}_k(\bar{S})$ there exists some $1 \leq i_J \leq t$ and some $J' \in \binom{J}{k_{i_J}}$ such that $J' \in \text{Cons}_{k_{i_J}}(\bar{S}_{i_J})$. Hence by pigeonhole there is some $1 \leq i \leq t$ and some $\mathcal{D} \subseteq \text{Cons}_k(\bar{S})$, $|\mathcal{D}| \geq \frac{1}{t} |\text{Cons}_k(\bar{S})|$ such that for every $J \in \mathcal{D}$, $i_J = i$. By double counting as in (1), this implies that there is some $\alpha'_i = \alpha'(k, k_i, \alpha) > 0$ such that $|\text{Cons}_{k_i}(\bar{S}_i)| \geq \frac{\alpha \binom{n}{k}}{\binom{n-k_i}{k-k_i}} \geq \alpha'_i \binom{n}{k_i}$ holds for all n . By FHP_{*k_i*} for $\varphi_i(x, y_i)$, there is some $\beta_i = \beta_i(\alpha'_i) > 0$ and $R \subseteq B$, $|R| \geq \beta_i n$ such that $\bigwedge_{\bar{b} \in R} \varphi_i(x, b_i)$ is consistent, hence $\bigwedge_{\bar{b} \in R} \varphi(x, \bar{b})$ is consistent. Thus taking $\beta := \min\{\beta_i : 1 \leq i \leq t\}$ does the job.

For local FHP_{*k*}, note that if all $\bar{b} \in B$ have the same type, then for each $1 \leq i \leq t$, all of the elements in $\{b_i : \bar{b} \in B\}$ also have the same type, so the proof goes through.

(3) We prove it by induction on the length of $|x|$. Let $\varphi(x_1, x_2; y)$ be given, and assume that FHP holds for all formulas $\psi(x, y)$ with $|x| < |x_1| + |x_2|$. Fix some $\alpha > 0$. Let $k_1, k_2 \in \mathbb{N}$ be arbitrary, and let $k = k_1 \times k_2$. Fix an arbitrary set $A \subseteq M_y$ of size n , let $\mathcal{F} := \{\varphi(x_1, x_2; a) : a \in A\}$. Let $B := \binom{A}{k_2}$, $\psi(x_2; y_1, \dots, y_{k_2}) := \exists x_1 \bigwedge_{1 \leq i \leq k_2} \varphi(x_1, x_2; y_i)$ and let $\mathcal{F}' = \{\psi(x_1; \bar{a}) : \bar{a} \in B\}$.

Assume that $|\text{Cons}_k(\mathcal{F})| \geq \alpha \binom{n}{k}$. For every $S \in \binom{A}{k}$ pick some presentation of it as a disjoint union of k_1 -many subsets of S of size k_2 . This defines an injection from the set $\text{Cons}_k(\mathcal{F})$ to $\text{Cons}_{k_1}(\mathcal{F}')$, so $|\text{Cons}_{k_1}(\mathcal{F}')| \geq \alpha \binom{n}{k} \geq \frac{\alpha}{k^k} n^k$. We also have $|\mathcal{F}'| = \binom{n}{k_2} \leq \frac{n^{k_2}}{k_2!}$, and so

$$\left| \binom{\mathcal{F}'}{k_1} \right| \leq \binom{\frac{n^{k_2}}{k_2!}}{k_1} \leq \frac{\left(\frac{n^{k_2 k_1}}{(k_2!)^{k_1}} \right)}{k_1!} \leq \frac{1}{k_1! (k_2!)^{k_1}} n^k \leq \alpha' n^k$$

for some $\alpha' = \alpha'(k_1, k_2) > 0$ holds for all n . Combining we thus have that $|\text{Cons}_{k_1}(\mathcal{F}')| \geq \alpha'' \binom{|\mathcal{F}'|}{k_1}$ holds for some $\alpha'' = \alpha''(\alpha, k_1, k_2) > 0$ and all n . So, taking k_1 such that $\psi(x_2, \bar{y})$ satisfies FHP _{k_1} (exists by the inductive assumption), there is some $\beta' = \beta(k_1, \alpha'') > 0$ such that there is some $B_0 \subseteq B, |B_0| \geq \beta' |B|$ such that $\bigwedge_{\bar{a} \in B_0} \psi(x_2, \bar{a})$ is consistent, say realized by some $b \in M_{x_2}$. Now consider a new partitioned formula $\theta(x_1; x_2, y) := \varphi(x_1, x_2, y)$, and a set of parameters $A' = \{b\} \times A \subseteq M_{x_2 y}$ of size n , and a family $\mathcal{F}'' = \{\varphi(x_1; b, a) : a \in A\}$. By the choice of b we have $|\text{Cons}_{k_2}(\mathcal{F}'')| \geq \beta' |B| \geq \beta' \binom{n}{k_2} = \beta' \left| \binom{\mathcal{F}''}{k_2} \right|$. Taking $\alpha''' := \beta' > 0$ and k_2 such that $\theta(x_1; x_2, y)$ satisfies FHP _{k_2} (again, exists by the inductive assumption), there is some $\beta = \beta(\alpha''', k_2) > 0$ and some consistent subtuple $\mathcal{F}_0 \subseteq \mathcal{F}''$ of size $\geq \beta n$, say it is realized by c . But then the tuple $(c, b) \in M_{x_1 x_2}$ realizes the family $\mathcal{F}^* = \{\varphi(x_1, x_2; a) : \varphi(x_1, b, a) \in \mathcal{F}_0\}$ and $|\mathcal{F}^*| \geq \beta n$. Unwinding we have that the choice of β only depended on φ, α and k_1, k_2 , which shows that $\varphi(x_1, x_2; y)$ satisfies FHP _{k} , as wanted.

(4) Immediate from the definitions. \square

Remark 2.12. The bound in Lemma 2.11(2) is optimal. Consider the model companion of the theory of two linear orders \leq_1, \leq_2 . Let $\varphi_i(x; yy') := y \leq_i x \leq_i y'$. Then $\varphi_i(x; yy')$ satisfies FHP₂ for $i \in \{1, 2\}$, but it is not hard to see that $\varphi(x; y_1 y_2 y'_1 y'_2) = \bigvee_{i=1}^2 \varphi_i(x, y_i y'_i)$ does not by taking two families of disjoint intervals for each of \leq_i (it satisfies FHP₃, however).

We will use the following lemmas frequently.

Lemma 2.13. *Let \mathcal{F} be a family of subsets of X , and assume that there is some $d \in \mathbb{N}$ such that $|S| \leq d$ for all $S \in \mathcal{F}$. Then \mathcal{F} satisfies FHP₂.*

Proof. Fix $\alpha > 0$, and let $(S_i : 1 \leq i \leq n)$ be a tuple of sets from \mathcal{F} . Let $C := \{I \subseteq [n] : |I| = 2, \bigcap_{i \in I} S_i \neq \emptyset\}$ and assume that $|C| \geq \alpha \binom{n}{2}$. The by pigeonhole there must be some $i^* \in [n]$ such that $i^* \in I$ for at least $\frac{\alpha}{2} n$ of the sets $I \in C$. But then there must be some element $a \in S_{i^*}$ belonging to at least $\frac{\alpha}{2d} n$ of the sets $S_i, i \in [n]$, so take $\beta := \frac{\alpha}{2d} > 0$. \square

We will use the following fact from combinatorics (see e.g. the introduction of [Für83]).

Fact 2.14. [EK68] *For each $k \in \mathbb{N}$, there is $\gamma = \gamma(k) > 0$ satisfying the following (can take $\gamma = \frac{k!}{k^k}$).*

If \mathcal{F} is a set of k -element subsets of a set X , then there exist sets $X_1, \dots, X_k \subseteq X$ and $\mathcal{F}' \subseteq \mathcal{F}$ such that:

- (1) $X_i \cap X_j = \emptyset$ for all $1 \leq i < j \leq k$,
- (2) $|\mathcal{F}'| \geq \gamma |\mathcal{F}|$,
- (3) $|X_i \cap S| = 1$ for all $S \in \mathcal{F}'$ and $1 \leq i \leq k$.

We need a slightly more general version of it.

Lemma 2.15. *The conclusion of Fact 2.14 holds even if \mathcal{F} is a multiset (i.e. we allow repetitions of k -subsets from X).*

Proof. Let k be given. We take $\gamma > 0$ to be as given by the fact for $k + 1$.

Now let X be a set and let $\mathcal{F} = (S_i : 1 \leq i \leq n)$ be a tuple of k -subsets of X , possibly with repetitions. Let $X' \supseteq X$ be a new set obtained from X by adding a new element b_i for each $1 \leq i \leq n$. Let $S'_i := S_i \cup \{b_i\}$ be a $(k + 1)$ -element subsets of X' , and let $\mathcal{F}' = \{S'_i : 1 \leq i \leq n\}$, then $|\mathcal{F}'| = |\mathcal{F}|$ and all sets in \mathcal{F}' are pairwise distinct. By Fact 2.14, we find some disjoint sets $X_1, \dots, X_{k+1} \subseteq X'$ and some $I \subseteq [n]$ with $|I| \geq \gamma n$ such that for any $i \in I$ we have $|S'_i \cap X_j| = 1$ for all $1 \leq j \leq k + 1$. By pigeonhole, there is some $I' \subseteq I$, $|I'| \geq \frac{\gamma}{k+1} n$ and $1 \leq j^* \leq k + 1$ such that $b_i \in X_{j^*}$ for all $i \in I'$. Let $X'_j := X \cap X_j$. Then $(X'_j : 1 \leq j \leq k + 1, j \neq j^*)$, $\gamma' := \frac{\gamma}{k+1}$ and $\mathcal{F}_0 := \{S_i : i \in I'\}$ satisfy the conclusion. \square

The following is easy to verify from the definition of FHP:

Remark 2.16. Assume $\psi(x, z)$ has FHP, $g : \mathbb{M}^y \rightarrow \mathbb{M}^z$ is a definable function and $\rho(y)$ is an arbitrary formula. Then the formula $\varphi(x, y) := \psi(x, g(y)) \wedge \rho(y)$ is also FHP.

2.2. FHP in Shelah's classification. Next we discuss the position of FHP theories in Shelah's classification hierarchy [She90]. We recall the definition of some relevant tree properties, and refer to e.g. [Che14] or [CR16] for further details.

Definition 2.17. Suppose T is a complete theory and $\varphi(x; y) \in L$ is a partitioned formula in the language of T (with x, y tuples of variables).

- (1) $\varphi(x; y)$ has the *k -tree property* (k -TP) if there is a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in \mathbb{M} such that
 - for all $\eta \in \omega^\omega$, $\{\varphi(x; a_\eta|_\alpha) : \alpha < \omega\}$ is consistent,
 - for all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta \smallfrown \langle i \rangle}) : i < \omega\}$ is k -inconsistent.

We say that $\varphi(x, y)$ has the *tree property* (TP) if it has the k -tree property for some $k \in \omega$; otherwise we say that $\varphi(x, y)$ is NTP.

- (2) $\varphi(x; y)$ has the *tree property of the first kind* (TP₁) if there is a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in \mathbb{M} such that
 - for all $\eta \in \omega^\omega$, $\{\varphi(x; a_\eta|_\alpha) : \alpha < \omega\}$ is consistent,
 - for all $\eta \perp \nu$ in $\omega^{<\omega}$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
 Otherwise we say that $\varphi(x, y)$ is NTP₁.
- (3) $\varphi(x; y)$ has the *k -tree property of the second kind* (k -TP₂) if there is an array of tuples $(a_{\alpha, i})_{\alpha < \omega, i < \omega}$ in \mathbb{M} such that
 - for all functions $f : \omega \rightarrow \omega$, $\{\varphi(x; a_{\alpha, f(\alpha)}) : \alpha < \omega\}$ is consistent,
 - for all α , $\{\varphi(x; a_{\alpha, i}) : i < \omega\}$ is k -inconsistent.

We say that $\varphi(x, y)$ has the tree property of the second kind (TP₂) if it has the k -TP₂ for some $k \in \omega$; otherwise we say that $\varphi(x, y)$ is NTP₂.

(4) T has one of the above properties if some formula does modulo T .

Fact 2.18. [She90, III.7.7, III.7.11] (see also [Adl07, Section 4] or [KKS14, Theorem 6.6]) *A complete theory T has TP if and only if it has TP₁ or TP₂.*

Fact 2.19. (1) *A theory T is simple if and only if it is NTP. If T has TP, then some partitioned formula $\varphi(x, y) \in L$ with $|x| = 1$ has 2-TP (see e.g. [KP97, Wag00]).*

(2) *If T has TP₂, then some partitioned formula $\varphi(x, y) \in L$ with $|x| = 1$ has TP₂ [Che14, Theorem 2.9]. And if $\varphi(x, y)$ has TP₂, then for some $t \in \omega$, the formula $\psi(x; y_0, \dots, y_t) := \bigwedge_{i < t} \varphi(x, y_i)$ has 2-TP₂ [Che14, Lemma 3.2].*

(3) [CR16] *If T has TP₁, then some partitioned formula $\varphi(x, y) \in L$ with $|x| = 1$ has TP₁.*

Fact 2.20. (see e.g. [CH14, Lemma 3.9]) *If $\varphi(x, y)$ has k -TP₂, then working in \mathbb{M} , we can find an array $(a_{i,j} : i, j \in \mathbb{N})$ as in Definition 2.17(3) such that moreover $\text{tp}(a_{i,j})$ is constant for all $i, j \in \mathbb{N}$.*

Proposition 2.21. *If $\varphi(x, y)$ is locally FHP, then it is NTP₂.*

Proof. Assume that $\varphi(x, y)$ has d -TP₂ for some $d \in \mathbb{N}$. Let an array $(a_{i,j} : i, j \in \mathbb{N})$ in \mathbb{M}^y witnessing this be as given by Fact 2.20, with $q := \text{tp}(a_{i,j})$ constant.

Let $k \in \mathbb{N}$ be arbitrary, we show that $\varphi(x, y)$ does not satisfy local FHP _{k} (on $q(\mathbb{M})$). Let $\alpha := \frac{1}{k^k} > 0$, fix some $m \in \mathbb{N}$, let $A := \{a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq m\}$ and consider the family $\mathcal{F} = \{\varphi(x, a) : a \in A\}$. Let $n := |A| = km$. As for every $f : [k] \rightarrow [m]$, $\{\varphi(x, a_{i,f(i)}) : 1 \leq i \leq k\}$ is consistent, we have that $|\text{Cons}_k(\mathcal{F})| \geq m^k \geq \left(\frac{n}{k}\right)^k \geq \alpha \binom{n}{k}$. On the other hand, by pigeonhole for any $S \subseteq A$ of size $\geq (d-1)k + 1$ there is some $1 \leq i \leq k$ and some $1 \leq j_1 < \dots < j_d \leq m$ such that $a_{i,j_t} \in S$ for all $1 \leq t \leq d$, hence $\bigwedge_{a \in S} \varphi(x, a)$ is inconsistent by the choice of A .

This shows that for any $\beta > 0$, if we take m such that $(d-1)k + 1 < \beta n = \beta km$, then there is no consistent subset of \mathcal{F} of size $\geq \beta n$. \square

The following theorem of Matousek is very important for our discussion.

Fact 2.22. [Mat04] *Let \mathcal{F} be a family of subsets of X , and assume that $d \in \mathbb{N}$ is such that $\pi_{\mathcal{F}}^*(n) = o(n^d)$ as $n \rightarrow \infty$ (e.g. if $vc^*(\mathcal{F}) < d$). Then \mathcal{F} satisfies FHP _{d} .*

Recall that a formula $\varphi(x, y)$ has IP (independence property) if there is an infinite set A of $|x|$ -tuples and for any $I \subseteq A$, there is a $|y|$ -tuple b_I such that $\models \varphi(a, b_I)$ if and only if $a \in I$ for all $a \in A$. A formula is NIP if it does not have IP, and a theory T is NIP if all formulas are NIP. Hence:

Fact 2.23. [Mat04] *If $\varphi(x, y)$ is NIP (in particular, if $\varphi(x, y)$ is stable) then it is FHP.*

The class of *low* theories, and in particular of low simple theories, is investigated in [Bue99, Sha00].

Definition 2.24. A formula $\varphi(x, y)$ is *low* if there is some $k \in \mathbb{N}$ such that for any indiscernible sequence $(a_i : i \in \mathbb{N})$, $\{\varphi(x, a_i) : i \in \omega\}$ is consistent if and only if all of its subsets of size k are consistent. A theory is low if it implies that every formula is low.

It was observed that NIP formulas are low in [CK12, Remark 3.33]. More generally we have:

Proposition 2.25. *If $\varphi(x, y)$ is locally WFHP, then it is low.*

Proof. Assume that $\varphi(x, y)$ satisfies local WFHP_k , we show that then $\varphi(x, y)$ is low with the same k in Definition 2.24. Let $(a_i : i \in \mathbb{N})$ be an indiscernible sequence, and assume that every subset of $\mathcal{F} := \{\varphi(x, a_i) : i \in \mathbb{N}\}$ of size k is consistent. By indiscernibility, it is enough to show that for every $N \in \mathbb{N}$ there is a strictly increasing subsequence $(i_j \in \mathbb{N} : j < N)$ such that $\{\varphi(x, a_{i_j}) : 1 \leq j \leq N\}$ is consistent. Let p_0 and $\beta = \beta(p_0)$ be as given for $\varphi(x, y)$ by local WFHP_k (see Definition 2.9). Note that \mathcal{F} satisfies the (p_0, k) -property by assumption. Let $n \in \mathbb{N}$ be such that $\beta n \geq N$. Hence some subset of $\{\varphi(x, a_i) : i < n\}$ of size $\geq \beta n \geq N$ is consistent — as wanted. \square

Remark 2.26. DFHP_k for $\varphi(x, y)$ implies a slightly weaker condition than lowness: if $\varphi(x, b)$ does not k -divide, then it does not divide. This is equivalent to lowness in simple (or even resilient) theories, but is not known to be equivalent to lowness in NTP_2 (see Proposition 4.13 and Question 4.14 in [BYC14]).

Recall that *wncfp*, or *weak nfcf*, is a strengthening of lowness which characterizes elementarity of lovely pairs of simple theories [BYPV03, Vas05].

Definition 2.27. We say that T is *wncfp* if:

- (1) T is low, i.e. for every $\varphi(x, y) \in L$ there is some k_φ such that for any sequence $(b_i : i \in \omega)$, if $\{\varphi(x, b_i) : i \in \omega\}$ is k_φ -consistent, then it is not n -inconsistent for any $n \in \omega$.
- (2) For any $\varphi(x, y)$ and $\psi(y, z)$ in L there is a number $n = n(\varphi, \psi)$ such that for any $c \in \mathbb{M}^z$, if there is a k_φ -inconsistent family $\{\varphi(x, b_i) : i < n\}$ with $b_i \models \psi(x, c)$, then there is an infinite such family.

Fact 2.28. (1) [Vas05, Proposition 2.8] *If T is supersimple, of SU-rank 1, then T is wncfp.*

(2) [Vas05, Corollary 3.10] *If T is stable, then T is wncfp if and only if T is nfcf.*

Remark 2.29. (1) In a simple theory T , the following are equivalent (and (a) implies (b) is true in any theory):

- (a) T is wncfp;
- (b) T is low and $Q_{\varphi, \psi}(z)$ is (type-)definable for all $\varphi(x, y), \psi(y, z) \in L$, where $Q_{\varphi, \psi}(c)$ hold if $\varphi(x, b)$ does not divide over c for all $b \models \psi(y, c)$.
- (2) Wncfp implies elimination of \exists^∞ , namely $Q_{x=y, \psi(y, z)}(c)$ holds if and only if $\psi(y, c)$ defines a finite set.
- (3) Definition 2.27(2) can be rephrased as saying that given $\varphi(x, y), \psi(y, z)$ there is some n such that for any $c \in \mathbb{M}^z$, if the family $\{\varphi(x, b) : b \models \psi(y, c)\}$ satisfies the (n', k) -property for some $n' \in \omega$, then it already satisfies the (n, k) -property.

Example 2.30. Consider the theory T_{Pr} of the group $(\mathbb{Z}, +)$ expanded by a predicate for the primes. Assuming Dickson's conjecture in number theory, T_{Pr} is supersimple SU-rank 1, hence wncfp by Fact 2.28, locally FHP theory which is not FHP (see Section 6.2 for the details). Note also that by Fact 2.28(2), any stable theory

with fcp satisfies FHP and not wnfc, so there is no implication between FHP and wnfc in general.

Proposition 2.31. *If T is ω -categorical and locally FHP, then T is FHP and wnfc.*

Proof. Fix $\varphi(x, y) \in L$. By ω -categoricity, there are only finitely many types p_1, \dots, p_m in $S_y(\emptyset)$, and all of them are isolated, say by $\psi_1(y), \dots, \psi_m(y)$. Then $\varphi(x, y)$ is equivalent to $\bigvee_{1 \leq i \leq m} (\varphi(x, y) \wedge \psi_i(y))$. As $\varphi(x, y)$ is locally FHP by assumption, we have that $\varphi(x, y) \wedge \psi_i(y)$ is FHP for each $1 \leq i \leq m$, and as the class of FHP formulas is closed under disjunctions by Lemma 2.11(2), we conclude that $\varphi(x, y)$ is FHP. Also, it is a general fact that if an ω -categorical theory is low, then it is wnfc (as any invariant set over a finite number of parameters is definable, see e.g. [Pal12]). Hence local FHP implies wnfc by Proposition 2.25. \square

3. FHP RELATIVELY TO A CLASS OF MEASURES

3.1. Keisler measures. Let T be a complete L -theory. We will work in $M \models T$, and $\mathbb{M} \succ M$ is a saturated elementary extension. For any set $A \subseteq \mathcal{U}$, a *Keisler measure* over A in variables x is a finitely additive probability measure on the Boolean algebra $L_x(A)$ of A -definable subsets of \mathbb{M}^x . We denote the space of Keisler measures over A (in variables x) as $\mathfrak{M}_x(A)$. Every element of $\mathfrak{M}_x(A)$ is in unique correspondence with a regular Borel probability measure on the space of types $S_x(A)$, and we will routinely use this correspondence. We recall some notions from [Kei87, HPP08, HP11, HPS13, CS21, Gan19], and refer to [Sta16, Che18] for a survey.

Definition 3.1. Let $\mu \in \mathfrak{M}_x(\mathbb{M})$ and $A \subseteq \mathbb{M}$ a small subset.

- (1) μ is *A -invariant* if for any partitioned $L(M)$ -formula $\varphi(x; y) \in L$ and any $b, b' \in \mathbb{M}^y$, if $b \equiv_A b'$ then $\mu(\varphi(x; b)) = \mu(\varphi(x; b'))$.
- (2) Assume that μ is A -invariant and $\varphi(x; y) \in L(A)$. We define the map $F_{\mu, A}^\varphi : S_y(A) \rightarrow [0, 1]$ by $F_{\mu, A}^\varphi(q) = \mu(\varphi(x; b))$, where $b \models q$ (this is well-defined by A -invariance of μ).
- (3) μ is *Borel-definable* (respectively, *definable*) over A if μ is A -invariant and for any partitioned $L(A)$ -formula $\varphi(x; y)$, the map $F_{\mu, A}^\varphi$ is Borel-measurable (respectively, continuous).
- (4) μ is *finitely satisfiable in A* if for any $L(\mathbb{M})$ -formula $\varphi(x)$, if $\mu(\varphi(x)) > 0$ then $\mathbb{M} \models \varphi(a)$ for some tuple a in A .
- (5) μ is *dfs* over A if μ is both definable over A and finitely satisfiable in A .
- (6) Given $\bar{a} \in (\mathbb{M}^x)^{<\omega}$, with $\bar{a} = (a_1, \dots, a_n)$, the associated *average measure* $\text{Av}_{\bar{a}} \in \mathfrak{M}_x(\mathbb{M})$ is defined by

$$\text{Av}_{\bar{a}}(\varphi(x)) := \frac{|\{i \in [n] : \mathbb{M} \models \varphi(a_i)\}|}{n}$$

for any $\varphi(x) \in L_x(\mathbb{M})$.

- (7) μ is *finitely approximated*, or *fam*, over A if for any $\varphi(x; y) \in L$ and any $\varepsilon \in \mathbb{R}_{>0}$, there exists a finite tuple \bar{a} from A such that for any $b \in \mathbb{M}^y$, $\mu(\varphi(x; b)) \approx_\varepsilon \text{Av}_{\bar{a}}(\varphi(x; b))$. In this case, we call \bar{a} a (φ, ε) -*approximation* for μ .

We say that μ is *invariant* if it is invariant over some small model $M \prec \mathbb{M}$, and similarly for the other properties.

Remark 3.2. Note that if μ is fam and $\Delta(x, y)$ is a finite set of partitioned formulas, then for every $\varepsilon > 0$ there exists a finite tuple \bar{a} so that \bar{a} is a $(\varphi(x, y), \varepsilon)$ -approximation for μ for all $\varphi(x, y) \in \Delta$ simultaneously (by coding finitely many formulas into one using additional variables).

Definition 3.3. Assume $\mu \in \mathfrak{M}_x(\mathbb{M})$ is Borel-definable and $\nu \in \mathfrak{M}_y(\mathbb{M})$ arbitrary. We let $\mu \otimes \nu$ be the unique measure in $\mathfrak{M}_{xy}(\mathbb{M})$ such that for any $\varphi(x, y) \in L_{xy}(\mathbb{M})$, we have

$$(\mu \otimes \nu)(\varphi(x, y)) = \int_{S_y(A)} F_{\mu, A}^\varphi d(\widehat{\nu|_A}),$$

where μ is A -invariant and A contains all the parameters from φ , and $\widehat{\nu|_A}$ is the unique regular Borel probability measure on $S_y(A)$ extending the Keisler measure $\nu|_A$.

See e.g. [CG22, Section 3.1] for an explanation why this product is well-defined and its basic properties. We will often abuse the notation slightly and replace $\widehat{\nu|_A}$ with either $\nu|_A$ or simply ν when it is clear from the context, and sometimes write $F_{\mu, A}^\varphi$ as F_μ^φ . In general, \otimes need not be commutative/associative on Borel definable measures in arbitrary theories.

Fact 3.4. (1) [CGH23, Theorem 2.18] *Suppose μ, ν are definable and λ arbitrary. Then $\mu \otimes \nu$ is definable, and $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$.*
 (2) [CGH23, Theorem 5.16] *If μ is fim and ν is Borel definable, or μ is fam and ν is definable. Then $\mu \otimes \nu = \nu \otimes \mu$.*

3.2. FHP relative to a class of measures. We define a generalization of the FHP (Definition 2.1) relatively to a class of definable Keisler measures.

Definition 3.5. Let \mathcal{M} be a first-order \mathcal{L} -structure, $\mathbb{M} \succ M$ a saturated model. Let $\varphi(x, y) \in L$ be a partitioned formula, and let $\mathfrak{M} \subseteq \mathfrak{M}_y(\mathbb{M})$ be a class of definable Keisler measures. We say that $\varphi(x, y)$ satisfies FHP(d, α, β) *relatively to* \mathfrak{M} if for any $\mu \in \mathfrak{M}$, if $\mu^{\otimes d}(\exists x \bigwedge_{i \in [d]} \varphi(x, y_i)) > \alpha$ then $\mu(\varphi(a, y)) > \beta$ for some $a \in \mathbb{M}^x$. And $\varphi(x, y)$ satisfies FHP _{d} *relatively to* \mathfrak{M} if for every $\alpha \in \mathbb{R}_{>0}$ there is some $\beta \in \mathbb{R}_{>0}$ so that $\varphi(x, y)$ satisfies FHP(d, α, β) relatively to \mathfrak{M} . And $\varphi(x, y)$ satisfies FHP *relatively to* \mathfrak{M} if it satisfies FHP _{d} relatively to \mathfrak{M} for some $d \in \mathbb{N}$.

We note that the usual FHP property is equivalent to the FHP property with respect to the class of finitely supported measures:

Proposition 3.6. *The formula $\varphi(x, y)$ satisfies FHP _{d} (as in Definition 2.1) if and only if it satisfies FHP _{d} relatively to the class $\mathfrak{M}_y^{\text{fin}}(\mathbb{M})$ of all Keisler measures supported on finite subsets of \mathbb{M}^y .*

Proof. Assume that $\varphi(x, y)$ satisfies FHP _{d} , and fix some $\alpha > 0$. Let $\mu \in \mathfrak{M}_y(\mathbb{M})$ be concentrated on a finite $B \subseteq \mathbb{M}^y$. Say $B = \{b_1, \dots, b_n\}$ and $\mu(\{b_i\}) = r_i \in [0, 1]$ with $\sum_{i=1}^n r_i = 1$. Assume that $\mu^{\otimes d}(\exists x \bigwedge_{i \in [d]} \varphi(x, y_i)) > \alpha$, and let $\varepsilon \in \mathbb{R}_{>0}$ be arbitrarily small. Then, wiggling the weights of the points a little bit one by one, we can choose a measure ν concentrated on B such that $\nu(\{b_i\}) = s_i$ with s_i rational, such that $\nu^{\otimes e}(C) \approx^\varepsilon \mu^{\otimes e}(C)$ for all $1 \leq e \leq d$ and all $C \subseteq (\mathbb{M}^y)^e$, in particular $\nu^{\otimes d}(\exists x \bigwedge_{i \in [d]} \varphi(x, y_i)) > \alpha - \varepsilon$.

Let $s_i = \frac{t_i}{D}$, where $t_i, D \in \mathbb{N}$ and D is a common denominator for the s_i 's (so $\sum_{i=1}^n t_i = D$). Then we choose a tuple of sets $\mathcal{S} = (S_i : 1 \leq i \leq D)$ such that it contains t_i repetitions of the set $\varphi(M, b_i)$, for each $1 \leq i \leq n$. Now from the assumption on the measure we have $\left| \left\{ (i_1, \dots, i_d) \in [D]^d : \bigcap_{1 \leq j \leq d} S_{i_j} \neq \emptyset \right\} \right| \geq (\alpha - \varepsilon) D^d$. Note that $\left| \left\{ (i_1, \dots, i_d) \in [D]^d : \bigvee_{1 \leq j < j' \leq d} i_j = i_{j'} \right\} \right| \leq \binom{d}{2} D^{d-1}$, so $\text{Cons}_d(\mathcal{S}) \geq \frac{1}{d^d} (\alpha - \varepsilon) D^d - \binom{d}{2} D^{d-1}$ (the factor $\frac{1}{d^d}$ is there since every set in $\binom{D}{d}$ is counted d^d times). Hence, taking $\varepsilon := \frac{\alpha}{2}$, we have $\text{Cons}_d(\mathcal{S}) \geq \frac{1}{2} \frac{1}{d^d} \frac{\alpha}{2} D^d$ for all D sufficiently large, so $\text{Cons}_d(\mathcal{S}) \geq \alpha' \binom{D}{d}$ for $\alpha' = \alpha'(\alpha, d) := \frac{1}{2} \frac{1}{d^d} \frac{\alpha}{2} > 0$. As $\varphi(x, y)$ satisfies FHP _{d} , there is some $\beta = \beta(\alpha')$ and $a \in \mathbb{M}^x$ such that, taking $C := \{i \in [D] : a \in S_i\}$, and $J := \{j \in [n] : \models \varphi(a, b_j)\}$, we have $|C| \geq \beta D$ (this is where we use that in the definition of FHP repetitions of sets are allowed). Note that if $a \in S_i$, then $a \in S_j$ for every copy of S_i appearing in \mathcal{S} as well, hence $\sum_{j \in J} t_j \geq \beta D$, and so $\nu(\varphi(a, y)) \geq \frac{\beta D}{D} \geq \beta$, as wanted.

Conversely, assume φ satisfies FHP _{d} relatively to $\mathfrak{M}_y^{\text{fin}}$ and we are given a tuple of sets $\mathcal{S} = \{\varphi(x, b_i) : 1 \leq i \leq n\}$ such that $\text{Cons}_d(\mathcal{S}) \geq \alpha \binom{n}{d}$. Define a finitely supported Keisler measure $\mu \in \mathfrak{M}_y^{\text{fin}}$ via $\mu(Y) := \frac{|\{i \in [n] : b_i \in Y\}|}{n}$. Then

$$\begin{aligned} \mu^{\otimes d} \left(\bigwedge_{i \in [d]} \varphi(x, y_i) \right) &= \frac{\left| \left\{ (i_1, \dots, i_d) \in [n]^d : \models \exists x \bigwedge_{j \in [d]} \varphi(x, b_{i_j}) \right\} \right|}{n^d} \\ &\geq \frac{\alpha \binom{n}{d}}{n^d} \geq \alpha' \end{aligned}$$

for $\alpha' := \frac{\alpha}{d^d} > 0$. Taking $\beta > 0$ as given for α' by FHP _{d} relatively to $\mathfrak{M}_y^{\text{fin}}$, we have that $\mu(\varphi(a, y)) \geq \beta$ for some $a \in \mathbb{M}^x$. Then $|\{i : \models \varphi(a, b_i)\}| \geq \beta n$, hence φ satisfies FHP _{d} . \square

This equivalence lifts further from finitely supported measures to fam measures (see Definition 3.1(7)). Namely, we have (by a repeated application of [CS21, Proposition 2.14(2)]; see also [CG20, Proposition 2.10]):

Lemma 3.7. *Let $\psi(x_1, \dots, x_d; z) \in L$ and $\varepsilon > 0$ be arbitrary. Assume that μ_i is a fam measure on M_{x_i} and $\bar{b}_i = (b_{i,1}, \dots, b_{i,n_i}) \in (\mathbb{M}^{x_i})^{n_i}$ is a $(\psi_i(x_i; z_i), \varepsilon)$ -approximation for μ_i , where $\psi_i(x_i; z_i)$ is obtained from $\psi(x_1, \dots, x_d, z)$ by partitioning its variables into two groups x_i and $z_i = x_1 \dots x_{i-1} x_{i+1} \dots x_d z$, for $i = 1, \dots, d$. Then $\bar{b} := (b_{i,j} : i \in [d], j \in [n_i])$ is a $(\psi(x_1, \dots, x_d; z), 2^d \varepsilon)$ -approximation for $\mu := \mu_1 \otimes \dots \otimes \mu_d$.*

In view of this, we can talk about FHP relatively to the class $\mathfrak{M}_y^{\text{fap}}$ of all fap measures on M_y with the product defined above. Note that $\mathfrak{M}_y^{\text{fin}} \subseteq \mathfrak{M}_y^{\text{fap}}$, and the products defined above coincide on $\mathfrak{M}_y^{\text{fin}}$.

Proposition 3.8. *A partitioned formula $\varphi(x, y) \in L$ satisfies FHP _{d} if and only if it satisfies FHP _{d} relatively to the class $\mathfrak{M}_y^{\text{fam}} \subseteq \mathfrak{M}_y(\mathbb{M})$ of all fam Keisler measures on \mathbb{M}^y .*

Proof. Assume that $\varphi(x; y) \in L$ satisfies FHP _{d} , and let $\alpha > 0$ be arbitrary. Let $\beta' > 0$ be as given for $\alpha' := \frac{\alpha}{2}$ by FHP _{d} relatively to $\mathfrak{M}_y^{\text{fin}}$ (using Proposition 3.6).

Let ε be arbitrary with $0 < \varepsilon < \min\{\frac{\alpha}{2}, \beta'\}$ and let $\beta := \beta' - \varepsilon > 0$ — can be chosen depending only on α . Let $\psi(y_1, \dots, y_d) := \exists x \bigwedge_{i \in [d]} \varphi(x, y_i)$. For $i \in [d]$, define $\psi_i(y_i; z_i) := \exists x \bigwedge_{i \in [d]} \varphi(x, y_i)$, with $z_i = y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_d$. Let

$$\Delta(y; z) := \{\psi_i(y_i, z_i) : 1 \leq i \leq d\} \cup \{\varphi(x; y)\},$$

where $z := x^{\wedge} z_i$. Now let $\mu \in \mathfrak{M}_y^{\text{fam}}$ be arbitrary with $\mu^{\otimes d}(\exists x \bigwedge_{1 \leq i \leq d} \varphi(x, y_i)) \geq \alpha$. By Remark 3.2, let $\bar{b} = (b_1, \dots, b_n) \in (\mathbb{M}^y)^n$ be an $\frac{\varepsilon}{2^d}$ -approximation for μ on all formulas in Δ simultaneously. Then, by Lemma 3.7, $\bar{b}' := ((b_{i_1}, \dots, b_{i_d}) : (i_1, \dots, i_d) \in [n]^d)$ is an ε -approximation for $\mu^{\otimes d}$ on $\psi(y_1, \dots, y_d)$. As $\text{Av}_{\bar{b}}^{\otimes d} = \text{Av}_{\bar{b}'}$, we thus have $\text{Av}_{\bar{b}}^{\otimes d}(\exists x \bigwedge_{1 \leq i \leq d} \varphi(x, y_i)) \geq \alpha - \varepsilon \geq \alpha'$. And $\text{Av}_{\bar{b}} \in \mathfrak{M}_y^{\text{fin}}(\mathbb{M})$, so by the choice of β' there is some $a \in \mathbb{M}^x$ so that $\text{Av}_{\bar{b}}(\varphi(a, y)) \geq \beta'$. As \bar{b} is also a (φ, ε) -approximation for μ , this implies $\mu(\varphi(a, y)) \geq \beta' - \varepsilon \geq \beta$.

The converse follows by Proposition 3.6 as $\mathfrak{M}_y^{\text{fin}} \subseteq \mathfrak{M}_y^{\text{fam}}$. \square

3.3. FHP for generically stable measures in NIP structures. Matousek's result in Fact 2.22 implies that every NIP formula $\varphi(x, y)$ satisfies FHP. In NIP, fam is one of the equivalent characterizations of *generically stable* measures (see Remark 5.19). Hence Proposition 3.8 combined with Matousek's theorem immediately implies the main theorem of [HPS12] (by taking the contrapositives and exchanging the roles of the variables in the statement).

Fact 3.9. [HPS12, Proposition 2.1] *Let T be NIP and let $\mu \in \mathfrak{M}_x(\mathbb{M})$ be a generically stable measure. For any formula $\varphi(x, y)$, if $\mu(\varphi(x, b)) = 0$ for all b , then there is some $d \in \mathbb{N}$ such that $\mu^{\otimes d}(\exists y(\varphi(x_1, y) \wedge \dots \wedge \varphi(x_n, y))) = 0$. Moreover, d depends only on φ .*

Conversely, under the global NIP assumption on the theory, we can quickly deduce from Fact 3.9 using compactness that every formula satisfies FHP. Indeed, fix $\varphi(x, y) \in L$ and let d be as given by Fact 3.9 for $\varphi(y, x)^* := \varphi(x, y)$. Assume $\varphi(x, y)$ does not satisfy FHP _{d} relative to $\mathfrak{M}_y^{\text{fam}}(\mathbb{M})$. Then there exists $\alpha > 0$ and for every $i \in \omega$ some $\mu_i \in \mathfrak{M}_y^{\text{fam}}(\mathbb{M})$ with $\mu_i^{\otimes d}(\exists x \bigwedge_{t \in [d]} \varphi(x, y_t)) \geq \alpha$ but $\mu_i(\varphi(a, y)) \leq \frac{1}{i}$ for all $a \in \mathbb{M}^x$. Following [Che25, Section 3.4], in an NIP theory T we identify the set of global fam measures $\mathfrak{M}_y(\mathbb{M})$ with a hyperdefinable set $\widetilde{\mathcal{M}}_y$, let $\mu \in \mathfrak{M}_y^{\text{fam}}(\mathbb{M}) \mapsto [\mu] \in \widetilde{\mathcal{M}}_y$ denote the bijection. The sets

$$X_i := \left\{ [\mu] \in \widetilde{\mathcal{M}}_y : \forall a \in \mathbb{M}^x \mu(\varphi(a, y)) \leq \frac{1}{i} \right\},$$

$$Y := \left\{ [\mu] \in \widetilde{\mathcal{M}}_y : \mu^{\otimes d} \left(\exists x \bigwedge_{t \in [d]} \varphi(x, y_t) \right) \geq \alpha \right\}$$

are type-definable (by [Che25, Remark 3.27, Proposition 3.29]). Note that $X_{i+1} \subseteq X_i$ and $[\mu_i] \in X_i \cap Y$. It follows by saturation of \mathbb{M} that there exists some $\mu \in \mathfrak{M}_x^{\text{fam}}(\mathbb{M})$ with $[\mu] \in Y \cap \bigcap_{i \in \omega} X_i$. That is, $\mu^{\otimes d}(\exists x \bigwedge_{t \in [d]} \varphi(x, y_t)) \geq \alpha$ and $\mu(\varphi(a, y)) = 0$ for all $a \in \mathbb{M}^x$. But this contradicts the choice of d .

Remark 3.10. We could instead use directly the folklore facts that, under NIP, ultralimits of fam measures are fam (as measures on the ultraproduct) and ultralimits commute with \otimes (see e.g. [Gan25]).

4. COLORFUL FRACTIONAL HELLY PROPERTY, BURDEN AND (p, q) -THEOREMS

In this section we consider a more general (“colorful”) version of the FHP property relatively to products of different measures and its relation to the model theoretic notion of burden (in particular strengthening and generalizing some of the results of Pillay from [Pil13]). We also observe that, at the level of the theory, the conclusion of the (p, q) -theorem is equivalent to NIP (equivalently, finite VC-dimension).

4.1. Colorful FHP, burden, VC-density. The following lemma is very similar to [CT20, Lemma 4.6] but in a slightly different setting:

Lemma 4.1. *Let $k \in \mathbb{N}$ and Keisler measures $\mu_i \in \mathfrak{M}_{x_i}(\mathbb{M})$ for $i \in [k]$ be definable and pairwise commuting (in particular each μ_i commutes with itself). Assume that $R \subseteq \mathbb{M}^{x_1} \times \dots \times \mathbb{M}^{x_k}$ is definable and such that $\bigotimes_{i \in [k]} \mu_i(R) \geq \alpha > 0$. For $d \in \mathbb{N}$, consider the definable set*

$$\Sigma_d := \left\{ (\bar{x}_1, \dots, \bar{x}_k) \in (\mathbb{M}^{x_1})^d \times \dots \times (\mathbb{M}^{x_k})^d : \right. \\ \left. (x_{1,i_1}, \dots, x_{k,i_k}) \in R \text{ for all } (i_1, \dots, i_k) \in [d]^k \right\}.$$

Then $\mu_1^{\otimes d} \otimes \dots \otimes \mu_k^{\otimes d}(\Sigma_d) \geq \alpha^{dk}$.

Proof. We will use Fact 3.4 and the pairwise commuting assumption freely. We have the following:

Claim 4.2. Assume $\mu \in \mathfrak{M}_x(\mathbb{M}), \nu \in \mathfrak{M}_y(\mathbb{M})$ are definable and pairwise commuting, and $E \subseteq \mathbb{M}^x \times \mathbb{M}^y$ is definable with $\mu \otimes \nu(E) \geq \alpha > 0$. Fix $d \in \mathbb{N}$ and let

$$\Gamma := \{(a; b_1, \dots, b_d) \in \mathbb{M}^x \times (\mathbb{M}^y)^d : (a, b_i) \in E \text{ for all } i \in [d]\}.$$

Then $\mu \otimes \nu^{\otimes d}(\Gamma) \geq \alpha^d > 0$.

Proof. Assume $M \prec \mathbb{M}$ is small so that μ, ν, E (and hence also Γ_d) are definable over M . Then, by Definition 3.3, $\mu \otimes \nu^{\otimes d}(\Gamma) = \int_{r \in S_x(M)} \nu^{\otimes d}(\Gamma_a) d\mu|_M(r)$, where $a \in \mathbb{M}^x$ with $a \models r$ is arbitrary and Γ_a denotes the fiber of Γ at a . Note that for each fixed $a \in \mathbb{M}^x$, $\Gamma_a = E(a, y_1) \wedge \dots \wedge E(a, y_d)$ belongs to the product Boolean algebra $L_{x_1}(\mathbb{M}) \times \dots \times L_{x_d}(\mathbb{M})$. As $\nu^{\otimes d}$ extends the product measure $\nu^{\times d}$, we have $\nu^{\otimes d}(\Gamma_a) = (\nu(E_a))^d$ for all $a \in \mathbb{M}^x$. Then, using Hölder inequality with $p = d, q = d/(d-1)$, we have $\mu \otimes \nu^{\otimes d}(\Gamma) = \int_{r \in S_x(M)} (\nu(E_a))^d d\mu = \int_{r \in S_x(M)} (\nu(E_a))^d d\mu \cdot \int_{r \in S_x(M)} 1^{\frac{d}{d-1}} d\mu \geq \left(\int_{r \in S_x(M)} \nu(E_a) d\mu \right)^d = (\mu \otimes \nu(E))^d \geq \alpha^d > 0$. \square

Now let $R \subseteq \mathbb{M}^{x_1} \times \dots \times \mathbb{M}^{x_k}$ be as given by assumption. For $i \in [k]$, let $V_i := \mathbb{M}^{x_i}$ and $U_i := \prod_{j < i} (\mathbb{M}^{x_j})^d \times \prod_{j > i} \mathbb{M}^{x_j}$, and consider the definable binary relation $E_i \subseteq U_i \times V_i$ given by $(\bar{x}_1, \dots, \bar{x}_{i-1}, x_{i+1}, \dots, x_k; x_i) \in E_i \iff (x_{1,j_1}, \dots, x_{i-1,j_{i-1}}, x_i, x_{i+1}, \dots, x_k) \in R$ for all $j_1, \dots, j_{i-1} \in [d]$. Let $\nu_i := \mu_1^{\otimes d} \otimes \dots \otimes \mu_{i-1}^{\otimes d} \otimes \mu_{i+1} \otimes \dots \otimes \mu_k$.

In particular μ_i, ν_i are pairwise commuting for every $i \in [k]$. Note that E_1 is equal to R , E_{i+1} is equal to Γ_i as obtained from E_i by the claim, that $\Gamma_k = \Sigma_d$ and $\mu_k \otimes \nu_k = \mu_1^{\otimes d} \otimes \dots \otimes \mu_k^{\otimes d}$ (up to a permutation and repartition of the variables). Hence, applying the claim repeatedly and using that all products of the measures involved commute, we have $\mu_{i+1} \otimes \nu_{i+1}(E_{i+1}) \geq \alpha^d \cdot \mu_i \otimes \nu_i(E_i)$

for all i . As $\mu_1 \otimes \nu_1 = \mu_1 \otimes \dots \otimes \mu_k$ and $\mu_1 \otimes \nu_1(E_1) \geq \alpha$, we conclude that $\mu_1^{\otimes d} \otimes \dots \otimes \mu_k^{\otimes d}(\Sigma_d) \geq \alpha^{dk}$. \square

Adler [Adl07] introduced *burden*, a notion based on the invariant κ_{inp} of Shelah [She90] which generalizes simultaneously dp-rank in NIP theories and weight in simple theories. For notational convenience we consider an extension Card^* of the linear order on cardinals by adding a new maximal element ∞ and replacing every limit cardinal κ by two new elements κ_- and κ_+ . The standard embedding of cardinals into Card^* identifies κ with κ_+ . In the following, whenever we take a supremum of a set of cardinals, we will be computing it in Card^* .

Definition 4.3. [Adl07] Let $p(x)$ be a (partial) type.

- (1) An inp-pattern of depth κ in $p(x)$ consists of $(\bar{a}_i, \varphi_i(x, y_i), k_i)_{i \in \kappa}$ with $\bar{a}_i = (a_{ij})_{j \in \omega}$ and $k_i \in \omega$ such that:
 - $\{\varphi_i(x, a_{ij})\}_{j \in \omega}$ is k_i -inconsistent for every $i \in \kappa$,
 - $p(x) \cup \{\varphi_i(x, a_{if(i)})\}_{i \in \kappa}$ is consistent for every $f : \kappa \rightarrow \omega$.
- (2) The *burden* of a partial type $p(x)$ is the supremum (in Card^*) of the depths of inp-patterns in it. We denote the burden of p as $\text{bdn}(p)$ and we write $\text{bdn}(a/A)$ for $\text{bdn}(\text{tp}(a/A))$.
- (3) By compactness, T is NTP₂ if and only if $\text{bdn}("x = x") < \infty$, if and only if $\text{bdn}("x = x") < |T|^+$.
- (4) A theory T is called *strong* if $\text{bdn}(p) \leq (\aleph_0)_-$ for every finitary type p (equivalently, there is no inp-pattern of infinite depth). Of course, if T is strong then it is NTP₂.

Fact 4.4. [Adl07]

- (1) Let T be NIP. Then $\text{bdn}(p) = \text{dprk}(p)$ for any p .
- (2) Let T be simple. Then the burden of p is the supremum of the weights of its complete extensions.

Fact 4.5. (1) [Che14] If T is any theory, then “burden +1” is sub-multiplicative, i.e. for any tuples a, b in \mathbb{M} and cardinals κ, λ we have that $\text{bdn}(a) < \kappa, \text{bdn}(b/a) < \lambda$ implies $\text{bdn}(a, b) < \kappa \times \lambda$.

- (2) [KOU13] If T is NIP, then burden (which is equal to dp-rank in this case) is sub-additive: for any tuples a, b , $\text{bdn}(a, b) \leq \text{bdn}(a) + \text{bdn}(b/a)$. By [Tak25], the same is true only assuming that T satisfies the dependent dividing conjecture (from [Che14]).
- (3) If T is simple, then burden is also sub-additive (by sub-additivity of weight in simple theories and Fact 4.4(2)).

It is open if burden is sub-additive in general (or in NTP₂) theories [Che14, Conjecture 2.7].

Proposition 4.6. Assume that $\text{bdn}(x) < k$ and let $\varphi_i(x, y_i) \in L$, $i \in [k]$, satisfy FHP relatively to a class of definable measures $\mathfrak{M}_i \subseteq \mathfrak{M}_{y_i}(\mathbb{M})$ (see Definition 3.5). Then for every $\alpha > 0$ there is $\gamma > 0$ satisfying the following.

Let $\mu_i \in \mathfrak{M}_i$ be such that μ_1, \dots, μ_k are pairwise commuting (in particular each μ_i commutes with itself). Assume that $\mu_1 \otimes \dots \otimes \mu_k \left(\exists x \bigwedge_{i=1}^k \varphi_i(x, y_i) \right) \geq \alpha$. Then there is some $i \in [k]$ and some $a \in \mathbb{M}^x$ such that $\mu_i(\varphi_i(a, y_i)) \geq \gamma$.

Proof. Let $\varphi_i(x, y_i) \in L$ and $\mu_i \in \mathfrak{M}_i$ for $i \in [k]$ be given, and let $\alpha > 0$ be arbitrary.

By Definition 4.3 and compactness, for every $d \in \mathbb{N}$ there is some $D = D(d) \in \mathbb{N}$ such that there is **no** rectangular array $(b_{i,j} \in \mathbb{M}^{y_i} : i \in [k], j \in [D])$ satisfying

$$(*) \quad \begin{cases} \{\varphi_i(x, b_{i,f(i)}) : i \in [k]\} \text{ is consistent for any } f : [k] \rightarrow [D], \\ \{\varphi_i(x, b_{i,j}) : j \in [D]\} \text{ is } d\text{-inconsistent for any } 1 \leq i \leq k. \end{cases}$$

By assumption φ_i satisfies FHP _{d_i} for some $d_i \in \mathbb{N}$, let $d := \max\{d_i : 1 \leq i \leq k\}$, and let $D = D(d)$ be as above.

For each $i \in [k]$, let $\psi_i(\bar{y}_i) := \bigwedge_{s \in \binom{[D]}{d}} (\neg \exists x \bigwedge_{j \in s} \varphi_i(x, y_{i,j}))$, where $\bar{y}_i = y_{i,1} \dots y_{i,D}$ with all of $y_{i,j}$ of the same sort as y_i . Let $\psi(\bar{y}) := \bigwedge_{1 \leq i \leq k} \psi_i(\bar{y}_i)$, where $\bar{y} = \bar{y}_1 \dots \bar{y}_k$.

Fix $\beta = \beta(\alpha, k, D) > 0$, to be determined later.

If $\mu_i^{\otimes d}(\exists x \bigwedge_{j \in [d]} \varphi_i(x, y_{i,j})) \geq \beta$ for some $i \in [k]$, then, as $\varphi_i(x, y_i)$ satisfies FHP _{d} relatively to \mathfrak{M}_i , there is some $\gamma_i = \gamma_i(\beta) > 0$ and some $a \in \mathbb{M}^x$ such that $\mu_i(\varphi(a, y_i)) \geq \gamma_i$. Then taking $\gamma := \min\{\gamma_i : i \in [k]\}$ we would be done.

So assume that $\mu_i^{\otimes d}(\exists x \bigwedge_{j \in [d]} \varphi_i(x, y_{i,j})) < \beta$ for all $i \in [k]$. Then, by the union bound,

$$\mu_i^{\otimes D}(\neg \psi_i(\bar{y}_i)) \leq \binom{D}{d} \mu_i^{\otimes d} \left(\exists x \bigwedge_{j \in [d]} \varphi_i(x, y_{i,j}) \right) \leq \binom{D}{d} \beta,$$

hence $\mu_1^{\otimes D} \otimes \dots \otimes \mu_k^{\otimes D}(\neg \psi(\bar{y})) \leq k \binom{D}{d} \beta$, and so $\mu_1^{\otimes D} \otimes \dots \otimes \mu_k^{\otimes D}(\psi(\bar{y})) \geq 1 - k \binom{D}{d} \beta$.

On the other hand, we are assuming that $\mu_1 \otimes \dots \otimes \mu_k(\exists x \bigwedge_{i=1}^k \varphi_i(x, y_i)) \geq \alpha$.

Let $\theta(\bar{y}) := \bigwedge_{(j_1, \dots, j_k) \in [D]^k} (\exists x \bigwedge_{1 \leq i \leq k} \varphi_i(x, y_{i,j_i}))$. Then applying Lemma 4.1 to $R(y_1, \dots, y_k) := \exists x \bigwedge_{i=1}^k \varphi_i(x, y_i)$ (noting that $\Sigma_d(\bar{y}) = \theta(\bar{y})$ in this case), we have that $\mu_1^{\otimes D} \otimes \dots \otimes \mu_k^{\otimes D}(\theta(\bar{y})) \geq \alpha^{Dk}$.

Let $\mu := \mu_1^{\otimes D} \otimes \dots \otimes \mu_k^{\otimes D}$. Then, if $k \binom{D}{d} \beta < \alpha^{Dk}$, we have $\mu(\psi(\bar{y}) \wedge \theta(\bar{y})) > 0$. But any realization $(b_{i,j} : i \in [k], j \in [D])$ of $\psi(\bar{y}) \wedge \theta(\bar{y})$ would satisfy $(*)$ — contradicting the choice of D .

Hence taking $\beta > 0$ sufficiently small compared to α, k and D , we see that $\gamma > 0$ above can be chosen depending only on $\varphi_1, \dots, \varphi_k$ and α , as wanted. \square

Remark 4.7. Conversely, assume $\text{bdn}(\mathbb{M}^x) \geq k$, witnessed by $\varphi_i(x, y_i)$, k_i and $\bar{a} = (a_{i,j})_{j \in \omega}$ for $i \in [k]$. Then for every $\gamma > 0$, taking $\mu_i \in \mathfrak{M}_{y_i}^{\text{fin}}(\mathbb{M})$ to be the measure supported on the finite set $\{b_{i,j} : j \in [n]\}$ with sufficiently large n so that $\gamma > k_i/n$, we have $\bigotimes_{i \in [k]} \mu_i(\exists x \bigwedge_{i=1}^k \varphi_i(x, y_i)) = 1$, but $\mu_i(\varphi(a, y_i)) < \gamma$ for all $i \in [k]$ and $a \in \mathbb{M}^x$. Hence Proposition 4.6 gives a measure-theoretic characterization of burden in FHP theories.

Remark 4.8. Proposition 4.6 generalizes and refines [Pil13] applied with $\mathfrak{M}_i = \mathfrak{M}^{\text{fam}}$, which in an NIP theory corresponds to the class of generically stable measures. But also, by Theorem 7.6, applies to pseudofinite fields with the class \mathfrak{M}_i given by localizing the ultralimit of counting measures to arbitrary definable sets.

A partitioned formula $\varphi(x, y) \in L$ has (dual) VC-density $\leq \ell \in \mathbb{R}$ if there exists $K \in \mathbb{R}$ such that, for all finite $B \subseteq \mathbb{M}^y$, $|S_\varphi(B)| \leq K|B|^\ell$ (note that $|S_\varphi(B)|$ is

equal to the dual shatter function $\pi_{\mathcal{F}_\varphi}^*$ for the family $\mathcal{F}_\varphi = \{\varphi(\mathbb{M}, b) : b \in \mathbb{M}^y\}$ of subsets of \mathbb{M}^x). And we let its VC-density $\text{vc}^*(\varphi)$ be the infimum of all such $\ell \in \mathbb{R}$. For a complete theory T and a tuple of variables x , we let $\text{vc}_T^*(x)$ denote the supremum of $\text{vc}^*(\varphi)$ over all formulas $\varphi(x, y) \in L$ (with x fixed and y arbitrary). The following observation appears in [ADH⁺16, KOU13, GH14]:

Fact 4.9. *For any theory T and tuple of variables x , $\text{bdn}(\mathbb{M}^x) \leq \text{vc}_T^*(x)$.*

It is a well-known open problem (stated in various variants in [ADH⁺16, KOU13, GH14]) if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ so that for any NIP theory T we have $\text{vc}_T^*(x) \leq f(\text{bdn}(\mathbb{M}^x))$ (one can take f to be linear in all known examples). Matousek’s theorem (Fact 2.22) demonstrates that the fractional Helly number of a formula $\varphi(x, y)$ is bounded by its dual VC-density $\text{vc}^*(\varphi)$. Proposition 4.6 implies that in fact it is bounded by the burden/dp-rank of \mathbb{M}^x :

Corollary 4.10. *In any FHP (so e.g. in NIP) theory T , the fractional Helly number of a formula $\varphi(x, y)$ is at most $\text{bdn}(\mathbb{M}^x) + 1$.*

Proof. Applying Proposition 4.6 with $\varphi_1 = \dots = \varphi_k := \varphi$ and arbitrary $\mu_1 = \dots = \mu_k \in \mathfrak{M}_y^{\text{fin}}$ (using Proposition 3.6). \square

The following “colorful” version of the fractional Helly property was proved for convex sets in [BFM⁺14] (see also [Kim15]):

Theorem 4.11. *Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be finite non-empty families of convex sets in \mathbb{R}^d , and assume that $\alpha \in (0, 1]$. If at least $\alpha|\mathcal{F}_1| \dots |\mathcal{F}_{d+1}|$ tuples of the form (S_1, \dots, S_{d+1}) , $S_i \in \mathcal{F}_i$, have a non-empty intersection, then some \mathcal{F}_i contains an intersecting subfamily of size $\beta|\mathcal{F}_i|$, with $\beta = \frac{\alpha}{d+1}$.*

Here “colors” correspond to several families of sets instead of one, and the usual FHP follows by taking all of these families to be equal to each other. Proposition 4.6 (combined with Fact 4.9) implies a generalization of Matousek’s theorem (Fact 2.22) to a colorful version:

Corollary 4.12. *Assume that \mathcal{F} is a family of sets with $\pi_{\mathcal{F}}^*(n) = o(n^d)$. Let $\alpha \in (0, 1]$. Then there is some β satisfying the following. Let $\mathcal{F}_1, \dots, \mathcal{F}_d \subseteq \mathcal{F}$ be finite non-empty families. If at least $\alpha|\mathcal{F}_1| \dots |\mathcal{F}_d|$ tuples of the form (S_1, \dots, S_d) , $S_i \in \mathcal{F}_i$, have a non-empty intersection, then some \mathcal{F}_i contains an intersecting subfamily of size $\beta|\mathcal{F}_i|$.*

4.2. (Colorful) (p, q) -theorem and finite VC-dimension. A celebrated result of Alon and Kleitman established a long standing conjecture of Hadwiger and Debrunner:

Fact 4.13. [AK92] *Let p, q, d be integers with $p \geq q \geq d + 1$. Then there exists an integer $N = N(d, p, q)$ such that: if \mathcal{F} is a finite family of convex subsets of \mathbb{R}^d satisfying the (p, q) -property (Definition 2.2), then there exists a set $A \subseteq \mathbb{R}^d$ with $|A| \leq N$ so that $A \cap S \neq \emptyset$ for every $S \in \mathcal{F}$.*

Their proof combines two earlier results about convex sets: the fractional Helly property [KL79] and the existence of (weak) ε -nets [ABFK92]. Using his result on the fractional Helly property for families of sets of finite VC-dimension and the existence of ε -nets [HW86], Matousek obtained an analog for families of finite VC-dimension:

Fact 4.14. [Mat04, Theorem 4] *Let p, q, d be integers with $p \geq q \geq d$. Then there exists an integer $N = N(d, p, q)$ such that: if \mathcal{F} is a finite family of sets with $\pi_{\mathcal{F}}^*(n) = o(n^d)$ satisfying the (p, q) -property, then there exists a set $A \subseteq \mathbb{R}^d$ with $|A| \leq N$ so that $A \cap S \neq \emptyset$ for every $S \in \mathcal{F}$.*

Matousek's (p, q) -theorem (Fact 2.22) played an important role in the study of NIP theories, starting with the proof of the uniform definability of types over finite sets (UDTFS) in NIP theories [CS13, CS15] (also in the study of definably amenable NIP groups [CS18]; and more recently in the proof of the *definable (p, q) -conjecture* of Chernikov and Simon in [Kap24]). In [BFM⁺14] the authors obtain a colorful (p, q) -theorem for convex sets, relying on their colorful fractional Helly property. Similarly, using the colorful fractional Helly property (Corollary 4.12), one can obtain a colorful (p, q) -theorem for families of sets of finite VC-dimension.

As we demonstrate in this paper, the class of structures in which all definable families of sets satisfy the fractional Helly property is much wider than the class of NIP structures. Here we point out however that at the level of the theory, the (p, q) -theorem characterizes NIP.

Definition 4.15. For $d \in \omega$, we say that a partitioned formula $\varphi(x, y) \in L$ is *d -pierceable* if for any $p \geq q \geq d$ there is some $N = N(p, q) \in \omega$ such that, taking $\mathcal{F} = \{\varphi(\mathbb{M}, b) : b \in \mathbb{M}^y\}$, if a finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ satisfies the (p, q) -property, then it admits a transversal of size at most N . A formula is pierceable if it is d -pierceable for some $d \in \omega$. A theory T is pierceable if every formula is pierceable.

So by Fact 4.14, every NIP theory is pierceable. Conversely, we have:

Proposition 4.16. *Assume that the formula $\varphi(x, y)$ is not NIP (i.e. it defines a family of sets of infinite VC-dimension). Then the formula $\psi(x; y_1, y_2) := \varphi(x, y_1) \wedge \neg\varphi(x, y_2)$ is not pierceable.*

Proof. As $\varphi(x, y)$ has IP, for any $m \in \omega$ there is $S \subseteq \mathbb{M}^y$ with $|S| = m$ so that: for any $S' \subseteq S$, there is some $e \in \mathbb{M}^x$ such that for all $c \in S$ we have $\models \varphi(e, c) \iff c \in S'$. Let $((a_i, b_i) : i < n)$, for $n = m^2 - m$, list **all** pairs in $S^2 \setminus \Delta$ (where $\Delta = \{(a, b) : a = b\}$ denotes the diagonal).

Claim 1. The family $\{\psi(\mathbb{M}; a_i, b_i) : i < n\}$ satisfies the $(4, 2)$ -property.

Let $\{(a'_i, b'_i) : i < 4\}$ be arbitrary pairs from $S^2 \setminus \Delta$ witnessing failure of the $(4, 2)$ -property. From pairwise inconsistency and assumption on S , for any $i \neq j < 4$ we must have $a_i = b_j$ or $a_j = b_i$. But this is easily seen to contradict the assumption $a_i \neq b_i$ for all $i < 4$.

Claim 2. For any $2 \leq q \in \omega$, there is some $p \in \omega$ such that the family $\{\psi(\mathbb{M}; a_i, b_i) : i < n\}$ satisfies the (p, q) -property (as long as n is sufficiently large with respect to p, q).

Fix $q \geq 2$. Let $\Gamma(\bar{y}, \bar{y}') := \{\exists x(\psi(x, \bar{y}) \wedge \psi(x, \bar{y}'))\}$. By Ramsey's theorem we can choose $p = p(q)$ large enough, such that every sequence (\bar{d}_i) from $S^2 \setminus \Delta$ of length p contains a Γ -indiscernible subsequence of length $q' := \max\{q, 4\}$.

Assume now that the family $\{\psi(\mathbb{M}; (a_i, b_i)) : i < n\}$ fails the (p, q) -property. That is, there is some $I \subseteq n$ with $|I| = p$ so that the family $\{\psi(x; a_i, b_i) : i \in I\}$ is q -inconsistent. By the choice of p , there is some $J \subseteq I$ with $|J| = q'$ so that the sequence $(a_i, b_i)_{i \in J}$ is Γ -indiscernible, and we still have that $\{\psi(x; a_i, b_i) : i \in J\}$ is inconsistent.

But by definition of ψ and assumption on S , this can only happen if already $\{\psi(x; a_i, b_i), \psi(x; a_j, b_j)\}$ is inconsistent for some $i \neq j \in J$. But then, by Γ -indiscernibility of $(a_i, b_i : i \in J)$, the set $\{\psi(x; a_i, b_i) : i \in J\}$ is 2-inconsistent. As $q' \geq 4$, it follows that $\{\psi(x; d_i) : i < n\}$ fails the $(2, 4)$ -property – contradicting Claim 1.

Claim 3. The family $\{\psi(\mathbb{M}; a_i, b_i) : i < n\}$ does not admit any transversal of size independent of n .

Fix k , and let m be sufficiently large, to be determined later. Assume that the family admits a transversal of size k . That is, we can choose a partition D_0, \dots, D_{k-1} of $S^2 \setminus \Delta$ such that each of the families $\{\psi(x; a_i, b_i) : (a_i, b_i) \in D_l\}$, $l < k$, is consistent.

This implies in particular that $\pi_{y_1}(D_l) \cap \pi_{y_2}(D_l) = \emptyset$ for each $l < k$, where π_{y_i} denotes the projection onto the corresponding coordinate (as otherwise we have some $(a_i, b_i), (a_j, b_j) \in D_l$ with $i \neq j$, such that $a_i = b_j$ — so $\psi(x; a_i, b_i) \wedge \psi(x; a_j, b_j)$ is inconsistent). We show that this is impossible.

Say $S = \{c_i : i < m\}$. For every $i < j < m$, $(c_i, c_j) \in D_l$ for some $l < k$. By Ramsey (assuming $m \gg k$) there is a subsequence $(c'_i : i < m')$ for $m' \gg k$, and some $l < k$ such that $(c'_i, c'_j) \in D_l$ for all $i < j < m'$. In particular (c'_0, c'_1) and (c'_1, c'_2) are both in D_l , thus $\pi_{y_1}(D_l) \cap \pi_{y_2}(D_l) \neq \emptyset$.

Combining Claims 2 and 3, we see that $\psi(x; y_1, y_2)$ is not d -pierceable for any $d \in \omega$. \square

Corollary 4.17. *Let T be a complete first-order theory. Then T is pierceable if and only if T is NIP.*

Remark 4.18. Note that this corollary cannot hold at the level of a formula. Indeed, let $M = (\mathbb{R}^2, P; E)$ with two sorts \mathbb{R}^2 and P consisting of all convex subsets of \mathbb{R}^2 , and $E \subseteq \mathbb{R}^2 \times P$ the membership relation.

It is well-known that the family of convex subsets of \mathbb{R}^2 has infinite VC-dimension (e.g. it shatter any finite subset of the unit circle), hence $E(x, y)$ is not NIP. However, by Fact 4.13, $E(x, y)$ is 3-pierceable.

Remark 4.19. Also, $E(x, y)$ in this structure is FHP (by the fractional Helly property for convex sets [KL79]), hence in particular $E(x, y)$ is NTP_2 (by Proposition 2.21). We observe however that the formula $\varphi(x; y_1, y_2) := E(x, y_1) \wedge \neg E(x, y_2)$ in this structure has 2- TP_2 .

Proof. Let \mathcal{F} denote the family of subsets of \mathbb{R}^2 of the form $A \cap (\mathbb{R}^2 \setminus B)$ with A, B convex. Let $D \subseteq \mathbb{R}^2$ be the (convex) closed unit disk. Since $\mathbb{N}^{<\mathbb{N}}$ (the set of all finite sequences of natural numbers) is countable and the border ∂D is a circle, we may choose a family of pairwise disjoint closed arcs $(I_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ on ∂D . For each $s \in \mathbb{N}^{<\mathbb{N}}$, let ℓ_s be the chord line joining the endpoints of I_s , and let H_s be the *closed* halfplane bounded by ℓ_s that contains the center of D . Define the corresponding (convex) cap $C_s := D \cap (\mathbb{R}^2 \setminus H_s)$. By choosing each chord ℓ_s sufficiently close to ∂D (equivalently, choosing each arc I_s sufficiently small), we may assume the caps $(C_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ are pairwise disjoint.

For $i, j \in \mathbb{N}$, define

$$F_{i,j} := \bigcup \{C_s : |s| > i \text{ and } s(i) = j\},$$

where $|s|$ is the length of s and $s(i)$ is its i th entry (defined only when $i < |s|$). We claim that each $F_{i,j}$ belongs to \mathcal{F} . Indeed, let

$$B_{i,j} := D \cap \bigcap \{ H_s : |s| > i \text{ and } s(i) = j \}.$$

Then $B_{i,j}$ is convex, being an intersection of convex sets (the disk D and halfplanes). And we have $F_{i,j} = D \cap (\mathbb{R}^2 \setminus B_{i,j}) \in \mathcal{F}$. We will use the parameters $y_{i,j} := F_{i,j}$ in the set sort P .

Fix $i \in \mathbb{N}$ and let $j \neq j' \in \mathbb{N}$. No finite sequence s can satisfy both $s(i) = j$ and $s(i) = j'$. Hence $F_{i,j}$ and $F_{i,j'}$ are unions of *disjoint* subfamilies of the pairwise disjoint caps (C_s), therefore $F_{i,j} \cap F_{i,j'} = \emptyset$. Equivalently, the row $\{\varphi(x; y_{i,j}) : j \in \mathbb{N}\}$ is 2-inconsistent.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any function, and for each $n \in \mathbb{N}$ let $s_n := (f(0), f(1), \dots, f(n-1)) \in \mathbb{N}^{<\mathbb{N}}$ be its initial segment of length n . Then for every $i < n$ we have $|s_n| = n > i$ and $s_n(i) = f(i)$, so by construction $C_{s_n} \subseteq F_{i,f(i)}$, hence $\bigcap_{i < n} F_{i,f(i)} \supseteq C_{s_n} \neq \emptyset$ for all $n \in \mathbb{N}$. Thus $\{\varphi(x; y_{i,f(i)}) : i \in \mathbb{N}\}$ is finitely satisfiable in M . \square

5. f -GENERICS AND FORKING IN FHP THEORIES

Several notions of largeness for definable sets (and their equivariant versions in definable group) play an important role in the model-theoretic study of tame classes of structures. In this section we recall several notions of “large”, or “generic” sets for definable groups studied in the literature, and discuss their relationship in FHP theories.

5.1. Notions of genericity.

Definition 5.1. An (abstract) group G is *amenable* if there is a left G -invariant finitely additive probability measure on the Boolean algebra of *all* subsets of G .

Fact 5.2. (see e.g. [Gar])

- (1) If G is amenable, then there is a bi-invariant finitely additive probability measure on $\mathcal{P}(G)$.
- (2) All (virtually) solvable groups are amenable. Non-abelian free groups are not amenable.

Definition 5.3. A definable group $G = G(\mathbb{M})$ is *definably amenable* if there is a left- $G(\mathbb{M})$ -invariant Keisler measure $\mu \in \mathfrak{M}_G(\mathbb{M})$ supported on G .

- Fact 5.4.**
- (1) If G is a definable group and T admits a model M so that $G(M)$ is amenable (as an abstract group), then G is definably amenable. (For (1)–(4) see [HPP08, Section 5].)
 - (2) If T is stable, then every definable group is definably amenable.
 - (3) Definably compact groups in o -minimal theories or in the p -adics (e.g. $SO_3(\mathbb{R})$) are definably amenable (and satisfy a stronger condition fsg, see also [Che25] for a discussion and references).
 - (4) The following NIP groups are not definably amenable: $PSL_2(\mathbb{R})$, $SL_2(\mathbb{Q}_p)$.
 - (5) If G is dp -minimal, then it is definably amenable ([Sto23], answering a question from [CPS14]).
 - (6) If G is pseudo-finite, then it is definably amenable (witnessed by the ultralimit of counting measures on finite groups whose ultraproduct is elementarily equivalent to G).

- (7) If T is small then every definable group is definably amenable (see [CHK⁺23, Corollary 4.14]).
- (8) There exist definable groups in simple theories that are not definably amenable ([CHK⁺23]).

Definition 5.5. Let $G = G(\mathbb{M})$ be a definable group, and $X = \varphi(\mathbb{M}) \subseteq G$ a definable subset, where $\varphi(x) \in L(\mathbb{M})$.

- (1) X is *generic* if there exist some finite $A \subseteq G$ so that $G = A \cdot X$.
- (2) X is *weakly generic* if there is some non-generic definable set $Y \subseteq G$ so that $X \cup Y$ is generic (equivalently, for some finite $A \subseteq G$, $G \setminus (A \cdot X)$ is not generic) [New09].
- (3) X is *non null*, i.e. there exists a left- G -invariant global Keisler measure $\mu \in \mathfrak{M}_G(\mathbb{M})$ supported on G so that $\mu(X) > 0$.
- (4) X is *f-generic* if there exists a small model $M \prec \mathbb{M}$ so that $g \cdot X$ does not fork over M for all $g \in G(\mathbb{M})$ [CS18].
- (5) X *does not G -divide* if there is no infinite sequence $(g_i)_{i < \omega}$ with $g_i \in G(\mathbb{M})$ and $k \in \mathbb{N}$ so that the family of sets $\{g_i \cdot X : i < \omega\}$ is k -inconsistent [CS18].

We considered the action of G on the left in each of the cases above. Similarly, we can consider the action on the right, or discuss bi-generics of different kinds.

We note that “generic” corresponds to “syndetic”, and “weak generic” corresponds to “piecewise syndetic” in the ergodic theory terminology. We summarize what is known about the relation between these notions of large definable sets, in general and in particular classes of theories (some examples below are translations of the standard examples/facts in the ergodic theory literature).

Fact 5.6. Let $G = G(\mathbb{M})$ be a definable group in $\mathbb{M} \models T$ and $X = \varphi(\mathbb{M}) \subseteq G$ for $\varphi(x) \in L$.

- (1) Any T .
 - (a) Complements of weakly generic sets and null sets are ideals in the Boolean algebra of definable sets (immediate from the definitions).
 - (b) X non-null $\Rightarrow X$ does not G -divide (see Theorem 5.11); generic \Rightarrow weak generic. If forking equals dividing over models (e.g. T is NTP_2), then f -generic is equivalent to non- G -dividing (see e.g. [MOS20, Lemma 3.7]).
 - (c) If G is amenable (as a discrete group), then weak generic \Rightarrow non-null (if X is weak generic, then there is some finite set $A \subseteq G$ so that: for every finite $B \subseteq G$, there is $g \in G$ with $Bg \subseteq AX$; then, given a right-almost-invariant Følner net $(B_i)_{i \in I}$ for G , can choose $g_i \in G$ with $B_i \cdot g_i \subseteq AX$; taking an ultralimit, we thus find a bi- G -invariant measure μ with $\mu(AX) = 1$, hence $\mu(X) \geq \frac{1}{|A|} > 0$).
- (2) Assume T is NIP.
 - (a) G is fsg [HPP08, HP11, HPS12] (so e.g. T is stable and G is arbitrary), or more generally G is definably amenable and admits a global generic type (see [CS18, Section 3.4]). Then G is definably amenable, and all notions (1)–(5) in Definition 5.5 are equivalent, and they are also equivalent to their counterpart under the right action of G .
 - (b) G is definably amenable if and only if G -dividing subsets of G form an ideal, if and only if non- f -generic subsets of G form an ideal (so

if and only if weak genericity is equivalent to either f -genericity or non- G -dividing) [Sto23].

(c) If G is definably amenable, then (2)–(5) in Definition 5.5 are equivalent [CS18, Theorem 1.2]; but depend on which side G acts, and (1) is strictly stronger in general (see [CS18, Proposition 6.3, Example 6.4]).

(d) Without assuming definable amenability, X generic $\not\Rightarrow$ X is non- G -dividing, even when T is NIP (consider the group $G = \mathrm{SL}_2(\mathbb{R})$, then

$$X := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : |a| \geq |c| \right\} \text{ is generic and } G\text{-dividing witnessed e.g. by}$$

$$g_i = \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix} \text{ for } i \in \mathbb{N}; \text{ see the discussion after Theorem 1.1 in [Sto23]}.$$

(3) Assume T is NTP_2 and G admits a global strongly f -generic type (e.g. T is simple and G arbitrary; or G is definably amenable [MOS20, Proposition 3.20]; or G admits a global type with a bounded orbit, see the proof of [CS18, Theorem 3.12] in the case of types).

(a) f -generic sets form an ideal, (4) and (5) are equivalent (see [MOS20, Proposition 3.10]), and weak generic \Rightarrow f -generic (see the proof of [CS18, Proposition 3.30]).

(b) Even for T simple and G amenable (as a discrete group), non G -dividing $\not\Rightarrow$ non-null (assuming Dickson's conjecture, see Proposition 6.11); and non-null $\not\Rightarrow$ weak generic (even assuming FHP additionally, see Remark 6.13).

Problem 5.7. Relaxing amenability: is it true that if G is a definably amenable group in an arbitrary theory and X is weakly generic, then X is non null?

5.2. Generics in amenable FHP groups. We would like to connect forking and f -generics with measures in FHP theories. We will use some standard facts about finitely additive probability measures.

Fact 5.8. [LM49] Let S be a set and $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{P}(S)$ be Boolean subalgebras. Let μ be a finitely additive probability measure on \mathcal{B}_0 . Then there is a finitely additive probability measure ν on \mathcal{B}_1 extending μ . Moreover, for any $X \in \mathcal{B}_1$ we can choose ν with $\nu(X) = r$ for any r satisfying

$$\sup \{ \mu(L) : L \in \mathcal{B}_0, L \subseteq X \} \leq r \leq \inf \{ \mu(U) : U \in \mathcal{B}_0, X \subseteq U \}.$$

We will use the following criterion for the existence of finitely additive probability measures due to Kelley (alternatively, we could use a version of this criterion for finitely supported probability measures:

Definition 5.9. Let \mathcal{B} be a Boolean algebra of subsets of X , and let $\mathcal{F} \subseteq \mathcal{B} \setminus \{\emptyset\}$. Given a finite sequence $\bar{S} = (S_1, \dots, S_n)$ of sets from \mathcal{F} , possibly with repetitions, let $i(\bar{S}) := \frac{k}{n}$, where k is the largest size of a subset $J \subseteq [n]$ such that $\bigcap_{i \in J} S_i \neq \emptyset$. We define the *intersection number* of \mathcal{F} as

$$i(\mathcal{F}) := \inf \{ i(\bar{S}) : \bar{S} \text{ is a finite sequence of sets from } \mathcal{F} \}.$$

Fact 5.10. [Kel59] Let $\mathcal{F} \subseteq \mathcal{B} \setminus \{\emptyset\}$ be given.

(1) Let μ be a finitely additive probability measure on \mathcal{B} and $\alpha > 0$ such that $\mu(S) \geq \alpha$ for all $S \in \mathcal{F}$. Then $i(\mathcal{F}) \geq \alpha$.

- (2) Conversely, if $i(\mathcal{F}) = \alpha > 0$, then there exists a finitely additive probability measure μ on \mathcal{B} such that $\mu(S) \geq \alpha$ for all $S \in \mathcal{F}$.

Theorem 5.11. *Assume T is weakly FHP (Definition 2.2) and NTP_2 (so for example if T is FHP, by Proposition 2.21) and $G = G(\mathbb{M})$ is an (\emptyset) -definable group. Assume that there exists a model M of T so that $G(M)$ is amenable (as a discrete group). Then for any $L(\mathbb{M})$ -definable set $X = \varphi(\mathbb{M}) \subseteq G(\mathbb{M})$, the following are equivalent:*

- (1) X is non- G -dividing;
- (2) X is f -generic;
- (3) X is non null.

Remark 5.12. We note that if G is an amenable group and $H \equiv G$ is elementarily equivalent to it (in the pure group language), then H is not necessarily amenable. Groups for which this holds are called *uniformly amenable* (see e.g. [DK18, Section 16.8] and references there; in particular [Wys88] gives an example of an amenable group that is not uniformly amenable).

Proof. (1) is equivalent to (2) by Fact 5.6(1b) and Fact 5.31, as T is NTP_2 . And (3) implies (1) is well known: if $\mu \in \mathfrak{M}_G(\mathbb{M})$ is left- $G(\mathbb{M})$ -invariant and $\mu(X) = \alpha > 0$, then $\mu(g \cdot X) = \alpha$ for all $g \in G(\mathbb{M})$ by left-invariance of μ , hence $i(\mathcal{F}) \geq \alpha$ for $\mathcal{F} := \{g \cdot X : g \in G(\mathbb{M})\}$ by Fact 5.10(1). Hence, if $(g_i)_{i \in \omega}$ is any indiscernible sequence in $G(\mathbb{M})$ and $n \in \omega$ is arbitrary, we get that $\bigcap_{i \in I} g_i \cdot X \neq \emptyset$ for some $I \subseteq [n]$ with $|I| \geq \alpha n$. As n was arbitrary, by indiscernibility this implies that $\{g_i \cdot X : i \in \omega\}$ is consistent, so X does not G -divide. So it remains to prove that (1) implies (3).

Assume $X = \varphi(\mathbb{M}, b)$, where $\varphi(x, y) \in L$ and b in \mathbb{M} , does not G -divide (fixed for the rest of the proof). Let $\varphi'(x; y, y') := \varphi((y')^{-1} \cdot x; y)$, and let k be such that $\varphi'(x; y, y')$ satisfies WFHP_k in T . Let $M \prec \mathbb{M}$ be small so that $G(M)$ is amenable, as a discrete group (we may assume that M is small by Löwenheim-Skolem, as subgroups of amenable groups are amenable).

It follows by saturation of \mathbb{M} that there is some $p \in \mathbb{N}$ (which we fix for the rest of the proof) such that the family of sets $\mathcal{F} := \{g \cdot X : g \in G(\mathbb{M})\}$ satisfies the (p, k) -property, meaning that for any $g_1, \dots, g_p \in G(\mathbb{M})$ (possibly with repetitions), there is some $I \subseteq [p]$ with $|I| = k$ so that $\bigcap_{i \in I} g_i \cdot X \neq \emptyset$. Indeed, if no such p existed, by saturation of \mathbb{M} we could extract an infinite indiscernible sequence $(g_i : i \in \omega)$ in $G(\mathbb{M})$ so that $\{g_i \cdot X : i \in \omega\}$ is k -inconsistent — contradicting non- G -dividing.

Now we work in M , and assume that b_0 in M is such that the family $\mathcal{F}_0 := \{g \cdot \varphi(M, b_0) : g \in G(M)\}$ satisfies the (p, k) -property. Let $\bar{S} = (S_1, \dots, S_n)$ be any finite sequence of sets from \mathcal{F}_0 , possibly with repetitions. As \mathcal{F}_0 satisfies WFHP_k (by the choice of k), and without loss of generality p is sufficiently large with respect to k , there is $\beta = \beta(p, k) > 0$ and $J \subseteq [n]$ with $|J| \geq \beta n$ and $\bigcap_{i \in J} S_i \neq \emptyset$. Hence $i(\bar{S}) \geq \beta$ (Definition 5.9). As β does not depend on the choice of \bar{S} , it follows that $i(\mathcal{F}_0) \geq \beta$.

Applying Fact 5.10(2) we find some finitely additive probability measure ν on $\mathcal{P}(G(M))$ such that $\nu(g \cdot \varphi(M, b_0)) \geq \beta$ for all $g \in G(M)$.

Now let ω be a bi-invariant finitely additive probability measure on $\mathcal{P}(G(M))$ given by amenability of $G(M)$ and Fact 5.2. For any $X \subseteq G(M)$, we define $\mu'(X) := \int_{G(M)} f_X(g) d\omega$, where $f_X(g) = \nu(g \cdot X)$. (Formally, we can work in M' the expansion of M naming all subsets of all of its powers, where $\nu, \omega \in \mathfrak{M}_G(M')$ are

definable Keisler measures over \emptyset , and let μ' be the definable convolution $\nu * \omega$, as studied in [CG22, CG23, CGK24].) It follows that μ' is a $G(M)$ -invariant finitely additive probability measure on $\mathcal{P}(G(M))$ and $\mu'(\varphi(M, b_0)) \geq \beta$. Letting $\mu \in \mathfrak{M}_G(M)$ be the restriction of μ' to definable sets, we get that μ is $G(M)$ -invariant and $\mu(\varphi(M, b_0)) \geq \beta$.

Now we want to lift this implication on the existence of appropriate measures from M to \mathbb{M} . We consider the following generalization of Construction (*) from [HPP08, Section 2]. Let $N = (M, P, \mathbb{R}; (E_\psi)_{\psi \in L})$ be the structure with sorts M (equipped with its full L -structure), P whose elements are the $G(M)$ -invariant Keisler measures in $\mathfrak{M}_G(M)$ (with no additional structure on it) and \mathbb{R} (equipped with the full structure of a real closed field), and for each $\psi(x, y) \in L$ the map $E_\psi : M^y \times P \rightarrow \mathbb{R}$ defined by $E_\psi(b, \mu) := \mu(\psi(x, b))$. Let $N^* = (M^*, P^*, \mathbb{R}^*)$ be a saturated elementary extension of N . In particular M^* , with its induced L -structure, is a saturated model of T — so we may identify it with \mathbb{M} ; and we have the standard part map $\text{st} : \mathbb{R}^* \cap [0, 1] \rightarrow \mathbb{R} \cap [0, 1]$. For any $m \in P^*$, $\psi(x, y) \in L$ and $b \in (M^*)^y$, we define $\mu_m(\psi(x, b)) := \text{st}(E_\psi(b, m)^{N^*})$. By definition of N and $N^* \succ N$, it follows that for every $m \in P^*$, $\mu_m \in \mathfrak{M}_G^L(M^*)$ is a $G(M^*)$ -invariant Keisler measure (but of course not all such measures have to appear as μ_m for some $m \in P^*$).

By the above we have that N satisfies the following sentence (with parameter β in M):

$$\forall b' \in M^y \left(\left(\forall g_1, \dots, g_p \in G \bigvee_{I \in \binom{[p]}{k}} \exists x \bigwedge_{i \in I} \varphi'(x; b', g_i) \right) \rightarrow \exists m \in P (E_\varphi(b', m) \geq \beta) \right).$$

Then, as $N^* \succ N$, the same sentence holds in N^* . As \mathcal{F} satisfies the (p, k) -property, it follows that $N^* \models \exists m \in P (E_\varphi(b, m) \geq \beta)$. This implies $\mu_m(\varphi(x, b)) \geq \beta > 0$, and $\mu_m \in \mathfrak{M}_G(\mathbb{M})$ is $G(\mathbb{M})$ -invariant (using the identification of M^* and \mathbb{M}) — as wanted. \square

Problem 5.13. Does the equivalence in Theorem 5.11 hold only assuming that G is *definably* amenable?

Remark 5.14. Let (X, \mathcal{F}) be a set system with X finite. A *fractional transversal* for \mathcal{F} is a function $\varphi : X \rightarrow [0, 1]$ such that for each $S \in \mathcal{F}$ we have $\sum_{x \in S} \varphi(x) \geq 1$. The *size* of a fractional transversal φ is $\sum_{x \in X} \varphi(x)$, and the *fractional transversal number* $\tau^*(\mathcal{F})$ is the infimum of the sizes of fractional transversals for \mathcal{F} .

We note that in our proof of Theorem 5.11 we could instead use the following result for finite families (which explicitly uses the duality of linear programming essentially equivalent to Kelley's criterion in the finite case) plus compactness:

Fact 5.15. [AKMM02] *For every d, p there exists $\alpha > 0$ such that: for any finite family \mathcal{F} satisfying FH($d + 1, \alpha, \beta$) with some $\beta > 0$ and the $(p, d + 1)$ -property, we have $\tau^*(\mathcal{F}) \leq T$ for some $T = T(p, d, \beta)$.*

5.3. Generics in fim FHP groups.

Definition 5.16. [HPP08] A definable group G is *fsg* (*finitely satisfiable generics*) if there is some $p \in S_G(\mathbb{M})$ and small $M \prec \mathbb{M}$ such that for every $g \in G(\mathbb{M})$, $g \cdot p$ is finitely satisfiable in $G(M)$.

Fact 5.17. [HPP08, Proposition 4.2] *Let T be any theory and $G = G(\mathbb{M})$ a definable fsg group, witnessed by some $p \in S_G(\mathbb{M})$. Then p is a (two-sided) generic type (i.e. $\varphi(x) \in p$ implies $\varphi(\mathbb{M})$ is both left and right generic in G).*

In [CGK24, Section 3.6], a generalization of fsg groups from NIP theories to fim groups in arbitrary theories was proposed, demonstrating that part of the theory of fsg groups in NIP theories survives:

Definition 5.18. [HPS13] Let $\mu \in \mathfrak{M}_x(\mathbb{M})$ and $M \prec \mathbb{M}$ a small model. A Borel-definable measure μ is *fim* (a *frequency interpretation measure*) over M if μ is M -invariant and for any \mathcal{L} -formula $\varphi(x, y)$ there exists a sequence of formulas $(\theta_n(x_1, \dots, x_n))_{1 \leq n < \omega}$ in $\mathcal{L}(M)$ such that:

- (1) for any $\varepsilon > 0$, there exists some $n_\varepsilon \in \omega$ satisfying: for any $k \geq n_\varepsilon$, if $\mathbb{M} \models \theta_k(\bar{a})$ then

$$\sup_{b \in \mathcal{U}^y} |\text{Av}(\bar{a})(\varphi(x, b)) - \mu(\varphi(x, b))| < \varepsilon;$$

- (2) $\lim_{n \rightarrow \infty} \mu^{(n)}(\theta_n(\bar{x})) = 1$.

We say that μ is fim if μ is fim over some small $M \prec \mathbb{M}$.

Remark 5.19. In NIP theories, fim is equivalent to each of the following two properties for measures: *dfs* (definable and finitely satisfiable) and *fam*, recovering the usual notion of generic stability for Keisler measures [HPS13]. Outside of the NIP context, fim (properly) implies fam over a model, which in turn (properly) implies dfs (see [CG20, CGH23]; and [CGK24, Section 3] for further properties of fim measures).

Fact 5.20. [CGH23] *In any theory, if μ is fim, $n \in \omega$ and $\sigma : [n] \rightarrow [n]$ is a permutation, then $\bigotimes_{i \in [n]} \mu_{x_i} = \bigotimes_{i \in [n]} \mu_{x_{\sigma(i)}}$.*

Definition 5.21. [CGK24, Definition 3.32] An (\emptyset) -definable group $G = G(\mathbb{M})$ is *fim* if there exists a left G -invariant fim measure $\mu \in \mathfrak{M}_G(\mathbb{M})$.

Remark 5.22. In any theory, if G is fim then it is both definably amenable and fsg. If T is NIP, G is fsg if and only if it is fim ([HPP08, HPS13]).

Fact 5.23. [CGK24, Proposition 3.33] *Suppose that $G = G(\mathbb{M})$ is a \emptyset -definable fim group, witnessed by a left- G -invariant fim measure $\mu \in \mathfrak{M}_G(\mathbb{M})$. Then μ is both the unique left G -invariant measure in $\mathfrak{M}_G(\mathbb{M})$ and the unique right G -invariant measure in $\mathfrak{M}_G(\mathbb{M})$.*

Lemma 5.24. *If $G = G(\mathbb{M})$ is fim, witnessed by $\mu \in \mathfrak{M}_G(\mathbb{M})$, then a definable set $X \subseteq G$ is generic if and only if $\mu(X) > 0$.*

Proof. If X is generic then $A \cdot X = G$ for some finite $A \subseteq G$, hence $\mu(a \cdot X) \geq \frac{1}{|A|} > 0$ for some $a \in A$, so $\mu(X) > 0$ by G -invariance of μ .

Conversely, assume $\mu(X) > 0$, and say $X = \varphi(\mathbb{M})$ for some $\varphi(x) \in L(\mathbb{M})$. Then there exists some type $p \in S(\mu)$ in the support of μ so that $\varphi(x) \in p$. As μ is fim over some small $M \prec \mathbb{M}$, it is in particular finitely satisfiable in M , hence $g \cdot p$ is also finitely satisfiable in M for every $g \in G(\mathbb{M})$. So p witnesses that G is fsg, hence p is a (two-sided) generic in G by Fact 5.17, and so is $\varphi(x)$. \square

Here we show that under the FHP assumption, all notions of genericity agree in fim groups (generalizing Fact 5.6(2a)):

Proposition 5.25. *Let $G = G(\mathbb{M})$ be a definable fim group in an FHP theory. Then all notions of genericity (1)–(5) in Definition 5.5 are equivalent for definable subsets of G .*

Proof. Every FHP theory is NTP₂ by Proposition 2.21, so f -generic is equivalent to non- G -dividing by Fact 5.6(1b), and weak generic implies f -generic by definable amenability, NTP₂ and Fact 5.6(3a). In view of Lemma 5.24, it remains to show that if $X \subseteq G$ is non-generic, then X is G -dividing.

We follow the proof of [HPS12, Proposition 3.2], with some modifications. Let $M \prec \mathbb{M}$ be a small model so that X is M -definable. Let $\mu \in \mathfrak{M}_G(\mathbb{M})$ be the fim G -invariant global Keisler measure, fim over M . Let $\varphi(x, y) \in L(M)$ denote the formula defining the set $\{(x, y) \in G \times G : y \in x \cdot X\}$, so for $b \in G$, $\varphi(x, b)$ defines the set $b \cdot X^{-1}$.

As X is non-generic, also $b \cdot X^{-1}$ is non-generic for all $b \in G$. By Lemma 5.24, $\mu(\varphi(x, b)) = 0$ for all $b \in \mathbb{M}^y$ (using that μ is supported on G). As T is FHP and μ is fim (so in particular fam), by Proposition 3.8 (with the roles of the variables exchanged) there exists some $k \in \omega$ so that $\mu_{x_1, \dots, x_k}^{\otimes k}(\exists y \bigwedge_{i \in [k]} \varphi(x_i, y)) = 0$. Then, for an arbitrary $n \in \omega$, we have $\mu_{x_1, \dots, x_n}^{\otimes n}(\bigvee_{I \in \binom{[n]}{k}} \exists y \bigwedge_{i \in I} \varphi(x_i, y)) = 0$ (as $(\mu^{\otimes n})|_{(x_i: i \in I)} = \bigotimes_{i \in I} \mu_{x_i}$ by Fact 5.20 and $\mu^{\otimes n}$ extends the product measure on the product Boolean algebra, so we are taking the measure of a union of finitely many sets of measure 0). Then, taking an arbitrary tuple in the measure 1 complement of this definable set, we find some $(g_i : i \in [n])$ in $G(\mathbb{M})$ so that $\{\varphi(g_i, y) : i \in [n]\}$ is k -inconsistent, and $\varphi(g_i, y)$ defines the set $g_i \cdot X$. By saturation of \mathbb{M} , we can thus find an infinite M -indiscernible sequence $(g_i)_{i \in \omega}$ in $G(\mathbb{M})$ so that $\{g_i \cdot X : i \in \omega\}$ is k -inconsistent, showing that X is G -dividing. \square

5.4. Forking in FHP theories. In this section we make some remarks on an analogous question, connecting measures and forking in FHP theories, with respect to the action of the automorphism group $\text{Aut}(\mathbb{M})$ rather than a definable group.

Definition 5.26. T is *amenable* if every type over \emptyset extends to a global $\text{Aut}(\mathbb{M}/\emptyset)$ -invariant measure [HKP22, HKP20]. And a set $A \subseteq \mathbb{M}$ is an *extension base* if every type $p \in S(A)$ extends to a global type non-forking over A . Note that if T is amenable, then \emptyset is an extension base (if $p \in S(\emptyset)$ and $\mu \in \mathfrak{M}(\mathbb{M})$ extends it and is \emptyset -invariant, then any $p' \in S(\mu)$ in the support of μ gives a non-forking extension of p ; and the converse holds when T is NIP).

Fact 5.27. (See the introduction of [CHK⁺23] and references there.)

- (1) In any theory, for any small set of parameters A , if $\mu(\varphi(x, b)) > 0$ for some $\text{Aut}(\mathbb{M}/A)$ -invariant Keisler measure $\mu \in \mathfrak{M}_x(\mathbb{M})$, then $\varphi(x, b)$ does not fork over A .
- (2) If T is stable, the converse holds as well (i.e. the forking and the universal measure 0 ideals coincide).

Fact 5.28. [CHK⁺23] *There exist a simple (SU rank 1) amenable theory T in which there is a formula $\varphi(x, b) \in L(\mathbb{M})$ so that $\varphi(x, b)$ divides over \emptyset , but $\mu(\varphi(x, b)) = 0$ for every $\text{Aut}(\mathbb{M}/\emptyset)$ -invariant measure $\mu \in \mathfrak{M}_x(\mathbb{M})$.*

Further examples with the same property which are additionally \aleph_0 -categorical were found in [Mar25].

Fact 5.29. [PS23]

- (1) There exist an NIP theory T and formula $\varphi(x, b) \in L(\mathbb{M})$ so that $\varphi(x, b)$ does not fork over \emptyset , but $\mu(\varphi(x, b)) = 0$ for every global \emptyset -invariant Keisler measure μ .
- (2) If T is amenable, the formula $\varphi(x, b)$ forks over \emptyset if and only if $\mu(\varphi(x, b)) = 0$ for all \emptyset -invariant global Keisler measures.

Combined with the results of Section 5.2, these motivate the following question:

Problem 5.30. Assume T is FHP and amenable, is forking over \emptyset equivalent to universal measure 0? In particular, do examples in [CHK⁺23, Mar25] have FHP?

We observe a weaker result in this direction. First we recall some facts about dividing and forking in NTP₂ theories. We write $a \perp_C b$ to denote that $\text{tp}(a/bC)$ does not fork over C .

Fact 5.31. [CK12] *Let T be an NTP₂ theory, and let A be an extension base.*

- (1) $\varphi(x, b)$ forks over A if and only if it divides over A .
- (2) For any b in \mathbb{M} and a small model M , there is an M -indiscernible sequence $(b_i : i \in \omega)$ with $b_0 = b$ such that for any formula $\varphi(x, y) \in L(M)$, $\varphi(x, b)$ divides over M if and only if $\{\varphi(x, b_i) : i \in \omega\}$ is inconsistent.

Proposition 5.32. *Let T be NTP₂ and DFHP (i.e. FHP for dividing, see Definition 2.9(2)). Then the following are equivalent for any small extension base A and b from \mathbb{M} :*

- (1) $\varphi(x, b)$ does not fork/divide over A ,
- (2) there is some global Keisler measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ non-forking over A (i.e. for any formula $\psi(x, b) \in L(\mathbb{M})$, if $\mu(\psi(x, b)) > 0$ then $\psi(x, b)$ does not fork over A) and some $\varepsilon > 0$ such that $\mu(\varphi(x, b')) \geq \varepsilon$ for all $b' \equiv_A b$.

Proof. (2) \Rightarrow (1) is by Fact 5.10(1).

(1) \Rightarrow (2): Assume that $\varphi(x, b)$ does not divide over A . Assume $\varphi(x, y)$ satisfies DFHP _{k} for $k \in \omega$. Then, as in particular $\varphi(x, b)$ does not k -divide, using DFHP _{k} and Kelley's criterion (Fact 5.10) as in the proof of Theorem 5.11, there exist $\varepsilon > 0$ and a global Keisler measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ such that:

$$(*) \quad \mu(\varphi(x, b')) \geq \varepsilon \text{ for all } b' \in \mathbb{M}^y \text{ with } b' \equiv_A b.$$

We want to find a global Keisler measure ν satisfying (*) and non-forking over A . Assume no such ν exists. As both of these sets of measures are closed in $\mathfrak{M}_x(\mathbb{M})$, and the space of global Keisler measures is compact, there must exist some finitely many formulas $\psi_i(x, c) \in \mathcal{L}(\mathbb{M})$, $1 \leq i \leq n$ each dividing over A and some $\alpha > 0$ such that for any ν satisfying (*) we must have $\bigvee_{1 \leq i \leq n} \nu(\psi_i(x, c)) > \alpha$.

As A is an extension base, let $M \supseteq A$ be a small model such that $M \perp_A c$, and let $(c_i : i \in \omega)$ with $c_0 = c$ be as given by Fact 5.31(2). Then for any $j \in \omega$ we must have $\bigvee_{1 \leq i \leq n} \mu(\psi_i(x, c_j)) > \alpha$ (if $\bigvee_{1 \leq i \leq n} \mu(\psi_i(x, c_j)) \leq \alpha$ for some $j \in \omega$, then as (*) is an $\text{Aut}(\mathbb{M}/A)$ -invariant condition, hence also $\text{Aut}(\mathbb{M}/M)$ -invariant, taking an M -automorphism of \mathbb{M} sending c_j to c and applying it to μ , we would get some measure ν still satisfying (*) and violating the assumption above).

Note that by the choice of M and left transitivity of forking (see Lemma [CK12]), $\psi_i(x, c)$ divides over M for all $1 \leq i \leq n$. On the other hand, there is some $1 \leq i \leq n$ and an infinite subsequence $I \subseteq \omega$ such that $\bigwedge_{j \in I} (\mu(\psi_i(x, c_j)) > \alpha)$ holds. By Fact 5.10(1), compactness and indiscernibility this implies that $\{\psi_i(x, c_j) : j \in \omega\}$

is consistent, hence $\psi_i(x, c)$ does not divide over M (by the choice of $(c_j)_{j \in \omega}$) — a contradiction. \square

Problem 5.33. Can (2) be strengthened to: there exists a *weakly invariant* over A measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ with $\mu(\varphi(x, b)) > 0$, where weakly invariant means that for every $\psi(x, c) \in L(\mathbb{M})$, if $\mu(\psi(x, c)) > 0$ then there exists some $\alpha > 0$ so that $\mu(\varphi(x, c')) \geq \alpha$ for all $c' \equiv_A c$.

Remark 5.34. We note that conversely, any low theory satisfying the implication (1) \Rightarrow (2) in Proposition 5.32 is DFHP. Indeed, assume $\varphi(x, y)$ is low, so that dividing for any instance of φ implies k -dividing. Then if $\varphi(x, b)$ does not k -divide, it does not divide, so there exists $\varepsilon = \varepsilon(\text{tp}(b)) > 0$ and measure μ with $\mu(\varphi(x, b')) \geq \varepsilon$ for all $b' \equiv b$. By Fact 5.10(1) this implies that $i(\{\varphi(x, b') : b' \equiv b\}) \geq \varepsilon$ — hence DFHP _{k} holds.

6. FHP IN SOME GENERIC EXPANSIONS OF $(\mathbb{Z}, +)$

6.1. $(\mathbb{Z}, +, \text{Sqf})$. In this section we consider the structure $(\mathbb{Z}, +, \text{Sqf})$, where Sqf is a unary predicate for the integers not divisible by a square of any prime number, and show that T_{Sqf} satisfies FHP. This theory is studied in [BT21], and we recall some notation and results from there.

Definition 6.1. [BT21, Section 2] Below x be a single variable and $z = (z_i : i \in [n])$, $z' = (z'_i : i \in [n'])$ denote arbitrary tuples of variables, and for $a \in \mathbb{Z}$ and p prime, let $v_p(n)$ denote the p -adic valuation.

- (1) For $m \in \mathbb{N}_{>0}$ let $P_m^{\mathbb{Z}} := \{a \in \mathbb{Z} : v_p(a) < 2 + v_p(m) \text{ for all } p\}$, so $P_1^{\mathbb{Z}} = \text{Sqf}$. And for p prime and $l \in \mathbb{Z}$, let $U_{p,l}^{\mathbb{Z}} := \{a \in \mathbb{Z} : v_p(a) \geq l\} = p^l \mathbb{Z}$.
- (2) We consider the expansion M of $(\mathbb{Z}, +, \text{Sqf})$ in the language

$$\mathcal{L} = \left\{ +, -, 1, (P_m)_{m \in \mathbb{N}_{>0}}, (U_{p,l})_{p \text{ prime}, l \in \mathbb{Z}} \right\},$$

and let $T_{\text{Sqf}} := \text{Th}_{\mathcal{L}}(M)$. Let $\mathbb{M} \succ M$ be a monster model.

- (3) An L -formula $\theta(z)$ is a p -condition, for p prime, if it is a Boolean combination of formulas of the form $t(z) \notin U_{p,l}$ for $t(z)$ an arbitrary L -term (i.e. a \mathbb{Z} -linear combination of the variables from z) and $l \in \mathbb{N}$ (a p -condition is trivial if the Boolean combination is the empty conjunction). An L -formula $\theta(z)$ which is a Boolean combination of formulas of the form $t(z) = 0$ with $t(z)$ an arbitrary L -term is called an *equational condition*.
- (4) An L -formula $\psi(x, z, z')$ is a *special formula* if it is of the form

$$\bigwedge_{p \in S} \theta_p(x, z, z') \wedge \bigwedge_{i=1}^n (kx + z_i \in P_m) \wedge \bigwedge_{i'=1}^{n'} (kx + z'_{i'} \notin P_m),$$

where $k \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{N}_{\geq 1}$ and $\theta_p(x; z, z')$ are p -conditions for some finite set of primes S . And $\psi(x, z, z')$ is a *positive special formula* if it is of the form $\bigwedge_{p \in S} \theta_p(x, z, z') \wedge \bigwedge_{i=1}^n (kx + z_i \in P_m)$.

- (5) Given a (positive) special formula $\psi(x, z, z')$ as in (4) and prime p , its *associated p -condition* $\psi_p(x, z, z')$ is the formula

$$\theta_p(x, z, z') \wedge \bigwedge_{i=1}^n (kx + z_i \notin U_{p, 2+v_p(m)})$$

- (where we let θ_p be the trivial p -condition for $p \notin S$). Then $\psi(x, z, z')$ implies $\psi_p(x, z, z')$ for all p (in T_{Sqf}).
- (6) A (positive) G -system is a (positive) formula $\psi(x, c, c')$ with $\psi(x, z, z')$ a (positive) special formula and $c \in \mathbb{M}^z, c' \in \mathbb{M}^{z'}$. It is *non-trivial* if, assuming $c = (c_i : i \in [n]), c' = (c'_j : j \in [n'])$, we have $c_i \neq c'_j$ for all $(i, j) \in [n] \times [n']$.
- (7) For a prime p , a (positive) G -system $\psi(x, c, c')$ is *p -satisfiable* if $\psi_p(x, c, c')$ is satisfiable for each prime p . By (5), if $\psi(x, c, c')$ is satisfiable, then it is p -satisfiable for all p .

Fact 6.2. [BT21, Theorem 2.14] *If a non-trivial G -system $\psi(x, c, c')$ is p -satisfiable for all prime p , then it has infinitely many solutions in \mathbb{M} .*

Fact 6.3. [BT21, Theorem 1.1 and Theorem 3.4] *The L -theory T_{Sqf} eliminates quantifiers and is supersimple, of SU -rank 1.*

Fact 6.4. [BT21, Lemma 3.1] *Let $\varphi(x, y)$ be a quantifier-free L -formula, with x a singleton and y a tuple of variables. The $\varphi(x, y)$ is equivalent, modulo T_{Sqf} , to a disjunction of quantifier-free L -formulas of the form*

$$\rho(y) \wedge \varepsilon(x, y) \wedge \psi(x, t(y), t'(y)),$$

where $\rho(y)$ is a quantifier-free L -formula, $\varepsilon(x, y)$ an equational condition, $\psi(x, z, z')$ a special L -formula with $n := |z|, n' := |z'|$, and $t(y), t'(y)$ are tuples of L -terms of length n and n' , respectively.

We will need a refinement of [BT21, Lemma 2.11] in the special case of positive special formulas:

Lemma 6.5. *Let $\psi(x, z, z')$ be a positive special formula. Then there is some $\varepsilon = \varepsilon(\psi) \in \mathbb{R}_{>0}$ satisfying the following.*

For any tuples c, c' in \mathbb{Z} such that $\psi(x, c, c')$ is a positive G -system which is p -satisfiable for all p , let

$$\Psi(t) := \{a \in \mathbb{Z} : \psi(a, c, c') \text{ holds and } 0 < a < t\}.$$

Then

$$|\Psi(t)| \geq \varepsilon t - \sum_{i=1}^n \left(\sqrt{|c_i|} + \sqrt{|kt + c_i|} \right) - 1$$

for all $t \in \mathbb{N}$ (where $n = |z|$ and $k \in \mathbb{Z} \setminus \{0\}$ is as in Definition 6.1 for ψ).

Remark 6.6. So the refinement is that the choice of ε depends only on the formula $\psi(x, z, z')$, and not on the parameters c, c' . We remark that no such choice is possible for general, not necessarily positive, special formulas — as the following example demonstrates¹. Take $\psi(x, z, z') := x + z \in P_1 \wedge x + z' \notin P_1$. If an element a satisfies $\psi(x, c, c')$, then in particular $a + c'$ must be divisible by p^2 for some prime p . Suppose $c' - c$ is a multiple of all squares of primes $< N$. Then such a p as above must be $> N$. But then the density of $\{a \in \mathbb{Z} : \models \psi(a, c, c')\}$ is at most $\sum_{p > N} \frac{1}{p^2}$, which can be arbitrarily close to 0 when N is large.

¹We thank Chieu Minh Tran for pointing this out.

Proof. We follow closely the proof of [BT21, Lemma 2.11] with $h = 1$, $s = 0$. Let a positive special formula

$$\psi(x, z, z') = \bigwedge_{p \leq B_0} \theta_p(x, z, z') \wedge \bigwedge_{i=1}^n (kx + z_i \in P_m)$$

be fixed, and let $B := \max(\{|k|, n, B_0\}) + 1$. Let l'_p be the largest value of l such that the predicate $U_{p,l}$ occurs in θ_p . Let $D := \prod_{p \leq B} p^{l'_p}$. Note that both B and D depend only on ψ .

By the proof of [BT21, Lemma 2.10] for any c, c' such that $\theta_p(\mathbb{Z}, c, c')$ is non-empty for all $p \leq B$, there is some $r \in \{0, 1, \dots, D-1\}$ such that $a \equiv_D r$ implies $\models \bigwedge_{p \leq B} \theta_p(a, c, c')$ for all $a \in \mathbb{Z}$.

For a prime p , we let $l_p := 2 + v_p(m)$, and fix $M > B$ to be determined later.

Given some c, c' as in the statement of the lemma, let r be as given by the previous paragraph. Define

$$\Psi_M(t) := \left\{ a \in \mathbb{Z} : (0 < a < t) \wedge (a \equiv_D r) \wedge \left(\bigwedge_{B < p \leq M} \bigwedge_{i=1}^n ka + c_i \not\equiv_{p^{l_p}} 0 \right) \right\}.$$

First we establish a lower bound on $|\Psi_M(t)|$. For a prime $p > B$, we have that $p > |k|$, so k is invertible mod p^{l_p} . Hence there are at least $p^{l_p} - n$ choices of $r_p \in \{1, \dots, p^{l_p}\}$ such that: for all $a \in \mathbb{Z}$, if $a \equiv_{p^{l_p}} r_p$ then $\models \bigwedge_{i=1}^n (ka + c_i \not\equiv_{p^{l_p}} 0)$. As $\{p^{l_p} : B < p \leq M\} \cup \{D\}$ are pairwise coprime, by the Chinese remainder theorem we have

$$|\Psi_M(t)| \geq \prod_{B < p \leq M} (p^{l_p} - n) \left\lfloor \frac{t}{D \prod_{B < p \leq M} p^{l_p}} \right\rfloor.$$

Namely, first we choose some $(r_p : B < p \leq M)$ as above, and independently for each such fixed choice count the elements in

$$\left\{ a \in \mathbb{Z} : 0 < a < t \wedge a \equiv_D r \wedge \left(\bigwedge_{B < p \leq M} a \equiv_{p^{l_p}} r_p \right) \right\}.$$

This implies

$$|\Psi_M(t)| \geq \frac{t}{D} \prod_{B < p \leq M} \frac{p^{l_p} - n}{p^{l_p}} \geq \frac{t}{D} \prod_{B < p \leq M} \left(1 - \frac{n}{p^{l_p}}\right) \geq \frac{t}{D} \prod_{B < p} \left(1 - \frac{n}{p^{l_p}}\right).$$

Set $\varepsilon := \frac{1}{2D} \prod_{B < p} \left(1 - \frac{n}{p^{l_p}}\right) > \prod_{B < p} \left(1 - \frac{1}{p^{3/2}}\right) > 0$ (using Euler's product formula, as $l_p \geq 2$ for all p by definition). Then $|\Psi_M(t)| \geq 2\varepsilon t$.

We may assume $k > 0$ (replacing c by $-c$ and c' by $-c'$ if necessary). If $a \in \Psi_M(t) \setminus \Psi(t)$, then $0 < a < t$, hence $c_i < ka + c_i < kt + c_i$ for all $i \in [n]$; and we must have $ka + c_i \notin P_m$ for some $i \in [n]$ (in particular $n \geq 1$), hence $ka + c_i$ is a multiple of p^{l_p} for some p by definition of P_m , so necessarily for some $p > M$ by definition of $\Psi_M(t)$.

For each p and $i \in [n]$, the number of non-zero multiples of p^{l_p} in the interval $(c_i, kt + c_i)$ is $\lfloor ktp^{-l_p} \rfloor + \gamma$ for some $\gamma \in \{0, 1, -1, -2\}$. If $\gamma = 1$, as $l_p \geq 2$ we moreover have $p^2 \leq |c_i|$ or $p^2 \leq |kt + c_i|$, so $p \leq \sqrt{|c_i|} + \sqrt{|kt + c_i|}$. As $l_p \geq 2$, we

have $\lfloor ktp^{-l_p} \rfloor \leq ktp^{-2}$. Hence we get

$$|\Psi_M(t) \setminus \Psi(t)| \leq t \sum_{p>M} \frac{nk}{p^2} + \sum_{i=1}^n \left(\sqrt{|c_i|} + \sqrt{|kt + c_i|} \right) + 1.$$

Note that taking M sufficiently large, depending only on D, B, n , and hence only on ψ , we get $\sum_{p>M} \frac{nk}{p^2} < \varepsilon$. Combining this with the lower bound estimate for $|\Psi_M(t)|$, we get the desired conclusion. \square

This gives a lower bound on the asymptotic density of the set of solutions of $\psi(x, c, c')$ which is independent of c and c' . Using this we have:

Corollary 6.7. *Let $\psi(x, z, z')$ be a positive special formula. Then there is some $\delta = \delta(\psi) > 0$ such that: for any $n \in \mathbb{N}$ and any tuples $(c_i, c'_i : i \in [n])$ with $\psi(x, c_i, c'_i)$ consistent for all i , there is some $I \subseteq [n]$, $|I| \geq \delta n$ such that $\{\psi(x, c_i, c'_i) : i \in I\}$ is consistent.*

Proof. Let $\delta := \frac{\varepsilon}{2} > 0$ with ε given by Lemma 6.5 for ψ . Let μ be a finitely additive probability measure on $\mathcal{P}(\mathbb{Z})$ defined, for all $X \subseteq \mathbb{Z}$, via $\mu(X) := \lim_{n \rightarrow \mathcal{U}} \frac{|X \cap [-n, n]|}{2n+1}$ for some fixed non-principal ultrafilter \mathcal{U} on \mathbb{N} . Let n and $(c_i, c'_i : i \in [n])$ be as in the statement. By Lemma 6.5, we have $\mu(\psi(\mathbb{Z}; c_i, c'_i)) \geq \delta$ for each $i \in [n]$. Then, by Fact 5.10(1), there exists $I \subseteq [n]$ with $|I| \geq \delta n$ so that $\bigcap_{i \in I} \psi(\mathbb{Z}; c_i, c'_i) \neq \emptyset$. \square

Theorem 6.8. *The theory T_{Sqf} is FHP (in particular $(\mathbb{Z}, +, \text{Sqf})$ is FHP, as a reduct). Moreover, every formula $\varphi(x, y)$ with $|x| \leq d$ satisfies FHP _{$d+1$} .*

Proof. To show FHP, by Lemma 2.11 it suffices to show that every partitioned formula $\varphi(x, y) \in L$ with $|x| = 1$ has FHP. By quantifier elimination (Fact 6.3) we may assume $\varphi(x, y)$ is quantifier-free. Then, using that the set of formulas satisfying FHP is closed under disjunctions (Lemma 2.11(2)), by Fact 6.4 we may assume that $\varphi(x, y)$ is of the form

$$\rho(y) \wedge \varepsilon(x, y) \wedge \psi(x, t(y), t'(y)),$$

where $\rho(y)$ is a quantifier-free L -formula, $\varepsilon(x, y)$ an equational condition, $\psi(x, z, z')$ a special L -formula with $s := |z|, s' := |z'|$, i.e.

$$\psi(x, z, z') = \bigwedge_{p \in S} \theta_p(x, z, z') \wedge \bigwedge_{i=1}^s (kx + z_i \in P_m) \wedge \bigwedge_{i'=1}^{s'} (kx + z'_{i'} \notin P_m)$$

where S is a finite set of primes, and $t(y), t'(y)$ are tuples of L -terms of length s and s' , respectively.

Then, by Remark 2.16, it is sufficient to show that every formula of the form

$$\varphi(x; \bar{z}) = \varepsilon(x, z'') \wedge \psi(x, z, z')$$

with $\bar{z} = (z, z', z'')$ is FHP. Again using that FHP is closed under disjunctions, we may assume that $\varepsilon(x, z'') = \bigwedge_{i=1}^v (t_i(x, z'') \square_i 0)$ with $v \in \mathbb{N}$, $\square_i \in \{=, \neq\}$ and t_i an L -term, i.e. a fixed \mathbb{Z} -linear combination of the variables from x, z'' .

By Fact 6.3, Fact 2.28 and Remark 2.29, T_{Sqf} eliminates \exists^∞ . That is, there is some $d = d(\varphi(x, \bar{z})) \in \mathbb{N}$ such that for any $\bar{c} \in M^{\bar{z}}$, $\varphi(M, \bar{c})$ is finite if and only if $|\varphi(M, \bar{c})| \leq d$. As the set of formulas with FHP is closed under disjunctions, and $\varphi(x, \bar{z}) = (\varphi(x, \bar{z}) \wedge |\varphi(x, \bar{z})| \leq d) \vee (\varphi(x, \bar{z}) \wedge |\varphi(x, \bar{z})| = \infty)$, we may treat each of these two disjuncts separately. Now the formula $\varphi(x, \bar{z}) \wedge |\varphi(x, \bar{z})| \leq d$ satisfies FHP₂ by Lemma 2.13, and we show that $\varphi(x, \bar{z}) \wedge |\varphi(x, \bar{z})| = \infty$ satisfies FHP₁.

So fix $\alpha > 0$, and assume that we are given some $(\bar{c}_i : i \in [n])$, $\bar{c}_i = c_i c'_i c''_i$ such that $\varphi(x, \bar{c}_i) \wedge |\varphi(x, \bar{c}_i)| = \infty$ holds for all $i \in I_0$ for some $I_0 \subseteq [n]$ with $|I_0| \geq \alpha n$. In particular, $\varphi(M, \bar{c}_i)$ is infinite for all $i \in I_0$.

For each $i \in [n]$, let A_i be the set of all elements appearing in the tuple c_i and B_i the set of all elements appearing in the tuple c'_i . Let $C_i := A_i \cup B_i$. Note that for $i \in I_0$, an element of C_i cannot simultaneously occur in both tuples c_i and c'_i as $\psi(x, c_i, c'_i)$ is consistent, so $C_i = A_i \sqcup B_i$. As $|A_i| \leq s$ and $|B_i| \leq s'$ for all $i \in I_0$, there is some $I_1 \subseteq I_0$ with $|I_1| \geq \frac{1}{ss'} |I_0| \geq \frac{\alpha}{ss'} n$ and $r \leq s, r' \leq s'$ so that $|A_i| = r, |B_i| = r', |C_i| = k := r + r'$ for all $i \in I_1$.

Applying Lemma 2.15 to the sequence $(C_i : i \in I_1)$ of k -element subsets of M , there is some $\gamma = \gamma(k) = \gamma(s, s') > 0$, $I_2 \subseteq I_1$ with $|I_2| \geq \gamma |I_1| \geq \frac{\alpha\gamma}{ss'} n$ and some pairwise disjoint subsets X_1, \dots, X_k of M such that $|C_i \cap X_j| = 1$ for all $i \in I_2, j \in [k]$. In particular, there is some $J_i \subseteq [k]$ with $|J_i| = r_1$ so that $A_i \subseteq \bigcup_{j \in J_i} X_j$ and $B_i \subseteq \bigcup_{j \in [k] \setminus J_i} X_j$. By pigeonhole we can fix J_i , and possibly permuting the index sets $[k]$, we find some $I_3 \subseteq I_2$ with $|I_3| \geq \frac{1}{\binom{k}{r_1}} |I_2| \geq \frac{\alpha\gamma}{ss'(s+s')^s} n$ such that for each $i \in I_3$, we have $A_i \subseteq X_1 \cup \dots \cup X_{r_1}$ and $B_i \subseteq X_{r_1+1} \cup \dots \cup X_k$. In particular, as X_1, \dots, X_k are pairwise disjoint, for all for all $i, j \in I_3$ we have $A_i \cap B_j = \emptyset$, so the tuples c_i and c'_j do not have any common elements.

Now consider the positive special formula

$$\psi'(x, z, z') := \bigwedge_{p \in S} \theta_p(x, z, z') \wedge \bigwedge_{i=1}^s (kx + z_i \in P_m).$$

As $\psi'(x, c_i, c'_i)$ is consistent for all $i \in I_3$, by Corollary 6.7 there is some $\delta = \delta(\psi') = \delta(\psi) > 0$ and some $J \subseteq I_3, |J| \geq \delta |I_3| \geq \frac{\alpha\gamma\delta}{ss'(s+s')^s} n$ such that the set $\{\psi'(x, c_i, c'_i) : i \in J\}$ is consistent.

Now consider the G -system $\tilde{\psi}(x; \tilde{c}, \tilde{c}') := \bigwedge_{i \in J} \psi(x, c_i, c'_i)$, where $\tilde{c} = (c_i : i \in J)$, $\tilde{c}' = (c'_i : i \in J)$ (and $c_i = (c_{i,j} : j \in [s]), c'_i = (c'_{i,j} : j \in [s'])$). This G -system is non-trivial by the choice of I_3 . For any prime p , its associated p -condition $\tilde{\psi}_p$ is

$$\bigwedge_{i \in J} \left(\theta_p(x, c_i, c'_i) \wedge \bigwedge_{j=1}^s (kx + c_{i,j} \notin U_{p, 2+v_p(m)}) \right),$$

and it is consistent as $\{\psi'(x, c_i, c'_i) : i \in J\}$ is consistent. Hence the whole G -system $\bigwedge_{i \in J} \psi(x, c_i, c'_i)$ has infinitely many solutions by Fact 6.2.

As $\varphi(x, \bar{c}_j)$ is infinite for all $j \in J$ by assumption, $\varepsilon(x, c''_j)$ is also infinite. As each formula $t_i(x, c''_j) = 0$ has at most one realization, we must have $\square_i = \text{“}\neq\text{”}$ for all $i \in [v]$.

As $\bigwedge_{i \in J} \psi(x, c_i, c'_i)$ is infinite, it contains infinitely many realizations outside of the finite set $\bigcup_{i \in [v]} \bigcup_{j \in J} \{a \in \mathbb{Z} : t_i(a, c''_j) = 0\}$. Hence $\{\varphi(x, \bar{c}_j) : j \in J\}$ has infinitely many realizations. Thus we have shown that the formula $\varphi(x, \bar{z}) \wedge |\varphi(x, \bar{z})| = \infty$ satisfies FHP(1, α, β) with $\beta := \frac{\alpha\gamma\delta}{ss'(s+s')^s} > 0$ chosen depending only on φ and α .

For the moreover part, we have that T_{Sqf} is simple, of SU rank 1 by Fact 6.3. Hence, using Fact 4.5(3), $\text{bdn}(\mathbb{M}^1) = 1$ and $\text{bdn}(\mathbb{M}^d) = d$ for all $d \in \mathbb{N}$, and we conclude by Corollary 4.10. \square

Problem 6.9. We expect that using the results in [BT21] this proof generalizes to show that $(\mathbb{Q}, +, \text{Sqf}, <)$ is FHP.

Problem 6.10. Is T_{Sqf} MS-measurable? (See Definition 7.1.)

6.2. $(\mathbb{Z}, +, \text{Pr})$. Let $T_{\text{Pr}} := \text{Th}_{\mathcal{L}}(\mathbb{Z}, +, \text{Pr})$ in the language

$$\mathcal{L} = (+, -, 0, 1, \text{Pr}, \text{Pr}_n, P_n : 2 \leq n < \omega),$$

where Pr is a predicate for the primes and their additive inverses, $P_n(x) \iff x \equiv 0 \pmod{n}$, $\text{Pr}_n(x) \iff P_n(x) \wedge \text{Pr}\left(\frac{x}{n}\right)$.

Proposition 6.11. T_{Pr} is not FHP.

Proof. The \emptyset -definable set Pr does not G -divide in the group $G := (\mathbb{Z}, +)$. This is a direct consequence of the following theorem of Maynard:

Fact 6.12. [May15, Theorem 1.2] *For every $k \in \mathbb{N}$ there exist $N = N(k) \in \mathbb{N}$ and $C = C(k) > 0$ satisfying the following. If $A = \{a_1, \dots, a_N\} \subseteq \mathbb{Z}$ is any set of N distinct integers, then*

$$\left| \left\{ \{h_1, \dots, h_k\} \subseteq A : \exists^\infty n \in \mathbb{Z} \bigwedge_{i=1}^k (n + h_i \text{ is prime}) \right\} \right| \geq C |\{ \{h_1, \dots, h_k\} \subseteq A \}|$$

This implies that for any $k \in \mathbb{N}$, there is no (indiscernible) sequence $(g_i)_{i < \omega}$ with $g_i \in G(\mathbb{M})$ so that $\{g_i + \text{Pr}(\mathbb{M}) : i \in \omega\}$ is k -inconsistent.

Assume that T_{Pr} was FHP. The group $G := (\mathbb{Z}, +)$ is amenable. It is well-known that the set of primes has upper Banach density 0, which implies that $\mu(\text{Pr}) = 0$ for any G -invariant finitely additive probability measure μ on $\mathcal{P}(G)$ (e.g. see [DNGJ⁺15, Section 2.2]). This contradicts Theorem 5.11. \square

Remark 6.13. We note the crucial difference with the case of square-free integers. Let $M = (\mathbb{Z}, +, \text{Sqf})$ be as in Section 6.1, let $G(M) := (\mathbb{Z}, +)$. By Fact 6.3 its theory $\text{Th}(M)$ is supersimple, of SU rank 1, and by Theorem 6.8 it is FHP. As classically Sqf has upper Banach density $\frac{\pi}{6}$, we have $\mu(\text{Sqf}(M)) \geq \frac{\pi}{6}$ for some left- $G(M)$ -invariant finitely additive probability measure on $\mathcal{P}(\mathbb{Z})$, hence the definable set Sqf is non-null in $G = G(\mathbb{M})$ (see Section 5.1).

But Sqf is not weakly generic. We use the standard argument that it is not piecewise syndetic. For $(p_n)_{n \in \omega}$ an increasing enumeration of the primes, let $r_n := p_0^2 \cdot \dots \cdot p_n^2$. For any $A = \{a_0, \dots, a_n\} \subset \mathbb{Z}$, by the Chinese Remainder Theorem every interval in \mathbb{Z} of length r_n contains a solution to the system $\{x \equiv a_i \pmod{p_i^2} : 0 \leq i \leq n\}$, so $G(M) \setminus (A \cdot \text{Sqf}(M))$ is generic. As $M \prec \mathbb{M}$, it follows that for every finite $A \subseteq G(\mathbb{M})$, the set $G(\mathbb{M}) \setminus (A \cdot \text{Sqf}(\mathbb{M}))$ is generic in $G(\mathbb{M})$ — hence $\text{Sqf}(\mathbb{M})$ is not weakly generic.

The following is a very strong number theoretic conjecture of Dickson:

Conjecture 6.14. (Dickson, 1904 [Dic04]) *Let $k \in \mathbb{N}_{\geq 1}$ and $\bar{f} = (f_i : i < k)$ where $f_i(x) = a_i x + b_i$ with a_i, b_i non-negative integers and $a_i \geq 1$ for all $i < k$. Assume that there does not exist any integer $n \geq 1$ simultaneously dividing all of the products $\prod_{i < k} f_i(s)$ for every non-negative integer s . Then there exist infinitely many natural numbers m such that $f_i(m)$ is prime for all $i < k$.*

Fact 6.15. [KS17] *Assuming Dickson's conjecture, T_{Pr} eliminates quantifiers and is supersimple, of SU-rank 1.*

In the rest of the section, we will prove the following theorem:

Theorem 6.16. (Assuming Dickson's conjecture) T_{Pr} is locally FHP.

In fact, we only require the types of singletons in the parameter tuples to be fixed along the sequence (as opposed to the types of whole parameter tuples). Namely, for every partitioned formula $\varphi(x, y) \in L$ there exists $d = d(\varphi)$ satisfying the following. Let $s := |y|$. For every $\alpha > 0$ there exists $\beta = \beta(\varphi, \alpha) > 0$ so that: for any $n \in \mathbb{N}$ and tuples $c_i = (c_{i,j} : j \in [s]), i \in [n]$ so that for each fixed $j \in [s]$, $\text{tp}(c_{i,j}) = \text{tp}(c_{i',j})$ for all $i, i' \in [n]$, if $|\{I \in \binom{[n]}{d} : \bigcap_{i \in I} \varphi(\mathbb{Z}, c_i) \neq \emptyset\}| \geq \alpha \binom{[n]}{d}$, then there exists $J \subseteq [n]$ with $|J| \geq \beta n$ so that $\bigcap_{i \in J} \varphi(\mathbb{Z}, c_i) \neq \emptyset$.

Given a partitioned formula $\varphi(x, y) \in L$, let $\beta^*(\varphi) > 0$ be a real number satisfying the following (assuming that it exists):

- (*) If $c_i = (c_{i,j} : j \in [s]), i \in [n]$ are tuples from \mathbb{Z} so that for each fixed $j \in [s]$, $\text{tp}(c_{i,j}) = \text{tp}(c_{i',j})$ for all $i, i' \in [n]$ and $\varphi(\mathbb{Z}, c_i)$ is infinite for all $i \in [n]$, then there is some $J \subseteq [n], |J| \geq \beta^* n$ such that $\bigcap_{i \in J} \varphi(\mathbb{Z}, c_i)$ is infinite.

In the following claims, we establish this property for increasing classes of formulas with $|x| = 1$.

Claim 6.17. Let $\varphi(x, y) \in \mathcal{L}$, where $|x| = 1$ and $y = (y_j)_{j \in [s]}$, be of the form

$$\bigwedge_{j=1}^s \text{Pr}(m_j x + y_j)^{t_j}$$

for some $m_j \in \mathbb{Z}$ and truth values $t_j \in \{0, 1\}$. Then there exists $\beta^*(\varphi) = \beta^*(s) > 0$ satisfying (*).

Proof. Let $n \in \mathbb{N}$ and tuples $(c_i : i \in [n])$ as in the assumption of (*) be given, with $c_i = (c_{i,j} : j \in [s])$.

We may assume that $m_j \geq 1$ for all $j \in [s]$. Indeed, if $m_j = 0$ then $\text{Pr}(c_{i,j})^{t_j}$ holds for all $i \in [n]$ by assumption, so we can ignore such conjuncts. Note that $a \models \text{Pr}(m_j x + c_{i,j})^{t_j} \Leftrightarrow a \models \text{Pr}((-m_j)x + (-c_{i,j}))^{t_j}$ for any $a, m_j, c_{i,j} \in \mathbb{Z}$. Hence, letting $J := \{j \in [s] : m_j < 0\}$, we could consider the formula $\varphi'(x, y)$ obtained from $\varphi(x, y)$ by replacing m_j by $-m_j$ for all $j \in J$, and the sequence $(c'_i : i \in [n])$ with c'_i obtained from c_i by replacing $c_{i,j}$ by $-c_{i,j}$ for all $j \in J$ (we still have that for every $j \in [n]$, $\text{tp}(c'_{i,j}) = \text{tp}(c_{i',j})$ for all $i, i' \in [n]$). So if the assumption holds for φ, c_i then it also holds for φ', c'_i ; and if β^* satisfies the conclusion for φ' , then it also satisfies the conclusion for φ .

For each $i \in [n]$, let C_i be the set of pairs $\{(m_j, c_{i,j}) : j \in [s]\}$. By pigeonhole, there exists some $I_0 \subseteq [n]$ with $|I_0| \geq \frac{1}{s}n$ so that $|C_i|$ is constant for all $i \in I_0$, denote it by k (so $k \leq s$).

By Lemma 2.15, we find some $\gamma = \gamma(s) > 0$, some $I_1 \subseteq I_0, |I_1| \geq \gamma n$ and some disjoint sets X_1, \dots, X_k such that $|C_i \cap X_j| = 1$ for all $i \in I_1$ and $1 \leq j \leq k$. Note that if for some $i \in [n]$ we have $(m_j, c_{i,j}) = (m_{j'}, c_{i,j'})$ for some $1 \leq j \neq j' \leq s$, then necessarily $t_j = t_{j'}$ as $\varphi(x, c_i)$ is consistent. Set $I_2^0 := I_1$, and by pigeonhole and induction on $1 \leq l \leq k$ we can choose $u_l \in \{0, 1\}$ and I_2^l such that

- $I_2^l \subseteq I_2^{l-1}$,
- $|I_2^l| \geq \frac{1}{2} |I_2^{l-1}|$,
- for any $i \in I_2^l$, if $(m, c) \in C_i \cap X_l$ and $(m, c) = (m_j, c_{i,j})$, then $t_j = u_l$ — i.e. either all occurrences of $\text{Pr}(mx + c)$ in $\varphi(x, c_i), i \in I_2^l$ are positive, or all are negative.

Let $I_2 := I_2^k$. Then $|I_2| \geq \frac{1}{2^k} |I_1| \geq \frac{\gamma}{2^s} n$ and, as the sets X_j 's are pairwise disjoint, we have that for any $i \neq i' \in I_2$ and $1 \leq j \neq j' \leq k$, if $\Pr(m_j x + c_{i,j}) \in \varphi(x, c_i)$ and $\neg \Pr(m_{j'} x + c_{i',j'}) \in \varphi(x, c_{i'})$, then $(m_j, c_{i,j}) \neq (m_{j'}, c_{i',j'})$.

Now Dickson's conjecture implies that the set $\bigcap_{i \in I_2} \varphi(\mathbb{Z}, \bar{c}_i)$ is infinite, hence taking $\beta^*(\varphi) := \frac{\gamma}{2^s} > 0$ shows that $\varphi(x, y)$ satisfies (*). Namely, letting $J^+ := \{j \in [s] : t_j = 1\}$, by [KS17, Lemma 2.3] and the choice of I_2 , as $m_j \geq 1$ for all $j \in [s]$ by assumption, it is sufficient to show that the assumption of Conjecture 6.14 holds for $\bar{f} = (f_{i,j} : i \in I_2, j \in J^+)$ where $f_{i,j}(x) = m_j x + c_{i,j}$. By [KS17, Remark 2.1] it suffices to show that for any fixed $N \in \mathbb{N}$, for every prime $r < N$, r does not divide $\prod_{(i,j) \in I_2 \times J^+} f_{i,j}(s)$ for all $s \in \mathbb{Z}$ simultaneously. That is, for every prime $r < N$, for some $0 \leq t < r$ we have $\bigwedge_{i \in I_2, j \in J^+} m_j t + c_{i,j} \not\equiv 0 \pmod{r}$. Assume this does not happen for r , and fix any $i^* \in I_2$. Then, as $c_{i,j} \equiv c_{i^*,j} \pmod{r}$ for all $i \in I_2, j \in J^+$ (since $\text{tp}(c_{i,j}) = \text{tp}(c_{i^*,j})$ by assumption), we get that for all $0 \leq t < r$, for some $j \in J^+$, $m_j t + c_{i^*,j} \equiv 0 \pmod{r}$.

But this implies that $\varphi(\mathbb{Z}, c_{i^*})$ is finite by [KS17, Remark 2.6], contradicting the assumption on c_{i^*} . \square

Claim 6.18. Assume $\varphi(x, y) \in L$, with $|x| = 1$ and $y = (y_j : j \in [s])$, is of the form

$$\bigwedge_{j \in S} \Pr(m_j x + y_j)^{t_j} \wedge \bigwedge_{j \in S'} P_{k_j}(m'_j x + y_j)^{t'_j}.$$

for some $S, S' \subseteq [s]$, $m_j, m'_j \in \mathbb{Z}$, $k_j \in \mathbb{N}_{\geq 2}$ and truth values $t_j, t'_j \in \{0, 1\}$. Then there exists $\beta^*(\varphi) = \beta^*(s) > 0$ satisfying (*).

Proof. Let $K = K(\varphi) := \prod \{k_j : j \in S'\}$, and let tuples $(c_i : i \in [n])$ as in the assumption of (*) be given, with $c_i = (c_{i,j} : j \in [s])$.

By pigeonhole, there are some $0 \leq r < K - 1$ and $I \subseteq [n], |I| \geq \frac{1}{K}$ such that for each $i \in I$, $\varphi(x, c_i) \wedge (x \equiv r \pmod{K})$ has infinitely many solutions in \mathbb{Z} . Note also that for any $a \in \mathbb{Z}$, if $a \equiv r \pmod{K}$, then $a \models \bigwedge_{j \in S'} P_{k_j}(m'_j x + c_{i,j})^{t'_j}$ for all $i \in I$.

Let $\varphi'(x', y')$ be the formula $\bigwedge_{j \in S} \Pr(\tilde{m}_j x' + y'_j)^{t_j}$, with $\tilde{m}_j := m_j K$, and let $c'_{i,j} := m_j r + c_{i,j}$. Then $\varphi'(x', c'_i)$ has infinitely many solutions for each $i \in I$ (as any a satisfying $\varphi(x, c_i) \wedge x \equiv r \pmod{K}$ gives $a' := \frac{a-r}{K}$ satisfying $\varphi'(x', c'_i)$, and $a_1 \neq a_2$ implies $a'_1 \neq a'_2$). And for any $i \in I$ and $a' \in \mathbb{Z}$ satisfying $\varphi'(x', c'_i)$, $a := K a' + r$ satisfies $\varphi(x, c_i)$ by the above. Note that for each $j \in S$ and assumption on c_i , all $(c'_{i,j} : i \in I)$ all have the same type. Hence we can take $\beta^*(\varphi) := \frac{1}{K} \cdot \beta^*(\varphi')$ satisfying (*), where $\beta^*(\varphi')$ exists by Claim 6.17. \square

Claim 6.19. Assume that $\varphi(x, y) \in L$ with $|x| = 1$ and $y = (y_j : j \in [s])$ is an arbitrary finite conjunction of formulas of the following form: $\Pr(mx + y_i)^t$, $\Pr_k(mx + y_i)^t$, $P_k(mx + y)^t$ where $m \in \mathbb{Z}$, $k \in \mathbb{N}_{\geq 2}$ and $t \in \{0, 1\}$ is a truth value. Then there is some $\beta^*(\varphi) > 0$ satisfying (*).

Proof. Assume first that $\varphi(x, y)$ contains a conjunct $\Pr_k(mx + y_j)$. Let tuples $(c_i : i \in [n])$ as in the assumption of (*) be given, with $c_i = (c_{i,j} : j \in [s])$.

For $r \in \{0, \dots, k-1\}$, let $\varphi_r(x, y) := \varphi(x, y) \wedge x \equiv r \pmod{k}$. By assumption and pigeonhole, there is some r and some $I \subseteq [n], |I| \geq \frac{1}{k} n$ such that $\varphi_r(x, c_i)$ is infinite for all $i \in I$.

We let $\varphi'_r(x'; y, y')$ be obtained from $\varphi(x, y)$ by replacing $\text{Pr}_k(mx + y_j)$ with $\text{Pr}(mx' + y')$ and replacing x by $kx' + r$ everywhere else.

Assume that $a \models \varphi_r(x, c_i)$. Then $a \equiv r \pmod{k}$, so $a = ka' + r$ for some a' . As $\models \text{Pr}_k(ma + c_{i,j}) \iff \models \text{Pr}_k(mka' + (mr + c_{i,j}))$, in particular $mr + c_{i,j}$ is divisible by k . Let $c'_i := \frac{mr + c_{i,j}}{k}$. Then

$$\models \text{Pr}_k(ma + c_{i,j}) \iff \models \text{Pr}_k(kma' + kc'_i) \iff \models \text{Pr}(ma' + c'_i).$$

Hence $\varphi'_r(x'; c_i, c'_i)$ has infinitely many solutions for all $i \in I$ (as $\varphi_r(x, c_i)$ has infinitely many solutions by assumption and a' above is uniquely determined by a), and for any $a' \in \mathbb{Z}$ and $i \in I$, if a' satisfies $\varphi'_r(x'; c_i, c'_i)$, then $a := ka' + r$ satisfies $\varphi(x, c_i)$. Finally, note that still all elements in any fixed coordinate of $(c_i \widehat{=} c'_i : i \in I)$ have the same type over \emptyset . Thus we can take $\beta^*(\varphi) := \frac{1}{k} \cdot \min\{\beta^*(\varphi'_r) : r \in \{0, \dots, k-1\}\} > 0$.

Iterating this procedure for at most s steps, we reduce to the case with no conjuncts of the form $\text{Pr}_k(mx + y_j)$. A similar analysis allows us to get rid of all conjuncts of the form $\neg \text{Pr}_k(mx + y)$, thus reducing to the case in Claim 6.18. \square

Claim 6.20. Every formula $\varphi(x, y) \in L$ with $|x| = 1$ satisfies local FHP, in the stronger form stated in Theorem 6.16.

Proof. Let $y = (y_i : i \in [s])$. The proof of Lemma 2.11(2) shows that the class of formulas satisfying local FHP in the strong form stated in Theorem 6.16 is closed under disjunctions. Using this and quantifier elimination in T_{Pr} (Fact 6.15), we may assume that $\varphi(x, y)$ is a conjunction of formulas of the form $\text{Pr}(mx + f(y))^t$, $\text{Pr}_k(mx + f(y))^t$, $P_k(mx + f(y))^t$ and $(mx + f(y) = 0)^t$ where $m \in \mathbb{Z}$, $k \in \mathbb{N}_{\geq 2}$, $t \in \{0, 1\}$ is a truth value and f is an L -term (i.e. a \mathbb{Z} -linear combination of the $y_j, j \in [s]$). Let $\varphi'(x; y, y')$ be obtained from $\varphi(x, y)$ by replacing each occurrence of $f(y)$ above with a new variable y' . As in Remark 2.16, it is easy to see that local FHP in the strong form stated in Theorem 6.16 for $\varphi'(x, y')$ implies the same for $\varphi(x, y)$. So we may assume that $\varphi(x, y)$ is a finite conjunction of formulas of the form $\text{Pr}(mx + y)^t$, $\text{Pr}_k(mx + y)^t$, $P_k(mx + y)^t$ and $(mx + y = 0)^t$.

By Fact 6.15, Fact 2.28 and Remark 2.29, T_{Pr} eliminates \exists^∞ . That is, there is some $D = D(\varphi) \in \mathbb{N}$ such that for any tuple c , $\varphi(\mathbb{Z}, c)$ is finite if and only if $|\varphi(\mathbb{Z}, c)| \leq D$.

The formula $\varphi(x, y) \wedge (\neg \exists^{>D} x \varphi(x, y))$ satisfies FHP_2 by Lemma 2.13. Hence it suffices to show the Theorem 6.16 holds for the formula $\psi(x, y) := \varphi(x, y) \wedge (\exists^\infty x \varphi(x, y))$ with $d = 1$.

Fix $\alpha > 0$, and let tuples $(c_i : i \in [n])$ with $c_i = (c_{i,j} : j \in [s])$ be so that for each j , $\text{tp}(c_{i,j}) = \text{tp}(c_{i',j})$ for all $i, i' \in [n]$ and $\psi(x, c_i)$ is consistent, hence $\varphi(\mathbb{Z}, c_i)$ is infinite, for all $i \in [n]$.

This implies in particular that $\varphi(x, y)$ cannot contain any conjuncts of the form $mx + y = 0$. Let $\varphi'(x, y)$ be obtained from $\varphi(x, y)$ by forgetting all of the conjuncts of the form $mx + y \neq 0$, then $\varphi'(x, y)$ is of the form considered in Claim 6.19.

Hence there is some $\beta^* = \beta^*(\varphi') > 0$ and $I \subseteq [n], |I| \geq \beta^*n$ such that the set $\bigcap_{i \in I} \varphi'(\mathbb{Z}, c_i)$ is infinite. In particular, it contains infinitely many elements outside of the *finite* set

$$\bigcup_{i \in I, j \in [s]} \{a \in \mathbb{Z} : ma + c_{i,j} = 0 \text{ for some conjunct } mx + y_j \neq 0 \text{ occurring in } \varphi(x, y)\}.$$

Then every such element satisfies $\bigwedge_{i \in I} \varphi(x, c_i)$, hence $\beta(\varphi) := \beta^* > 0$ shows that $\psi(x, y)$ satisfies local FHP₁ in the strong form stated in Theorem 6.16. \square

Finally it remains to observe that the proof of Lemma 2.11(3) goes through to show that in any theory, if every formula $\varphi(x, y)$ with $|x| = 1$ satisfies local FHP in the strong form stated in Theorem 6.16 (i.e. only singleton coordinates in the parameter tuples are required to have the same complete type), then every formula does.

Problem 6.21. We do not know if this is true for local FHP however (i.e. when the type of full parameter tuples is required to be the same).

7. MS-MEASURABLE STRUCTURES AND LARGE FINITE FIELDS SATISFY FHP

We recall the notion of an MS-measurable structure and some of its basic properties (see [MS08, EM08]).

Definition 7.1. An L -structure M is *MS-measurable* if for every non-empty set $X \subseteq M^n$ definable (with parameters), we have a pair $(\dim(X), \text{meas}(X)) \in \mathbb{N} \times \mathbb{R}_{>0} \cup \{(0, 0)\}$ satisfying the following properties:

- (1) For any partitioned L -formula $\varphi(x, y) \in L$ with $|x| = 1$, there is a finite set $D_\varphi \subseteq \mathbb{N} \times \mathbb{R}_{>0} \cup \{(0, 0)\}$ and finitely many L -formulas $\{\psi_{d,\mu}(y) : (d, \mu) \in D_\varphi\}$ partitioning M^y and such that for any $b \in M^y$, $\models \psi_{d,\mu}(b)$ if and only if $\dim(\varphi(M, b)) = d$ and $\text{meas}(\varphi(M, b)) = \mu$.
- (2) If X is finite, then $\dim(X) = 0$ and $\text{meas}(X) = |X|$.
- (3) If $X, Y \subseteq M^n$ are disjoint definable sets, then:

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\},$$

$$\text{meas}(X \cup Y) = \begin{cases} \text{meas}(X) + \text{meas}(Y) & \text{if } \dim(X) = \dim(Y), \\ \text{meas}(X) & \text{if } \dim(X) > \dim(Y), \\ \text{meas}(Y) & \text{if } \dim(X) < \dim(Y). \end{cases}$$

- (4) (“Fubini”) Let $f : X \rightarrow Y$ be a definable surjection such that $\dim(f^{-1}(a)) = d$, $\text{meas}(f^{-1}(a)) = r$ for all $a \in Y$. Then $\dim(X) = \dim(Y) + d$ and $\text{meas}(X) = r \text{meas}(Y)$.

Some of the main examples of MS-measurable structures are ultraproducts of finite fields (see below), finite simple groups of bounded Lie rank, vector spaces, etc. — we refer to [EM08, Example 2.4] for further examples.

Fact 7.2. [MS08] *Let M be an MS-measurable L -structure.*

- (1) *The condition (1) in Definition 7.1 holds for L -formulas $\varphi(x, y)$ with $|x|$ arbitrary [MS08, Proposition 5.7].*
- (2) *Any $M' \equiv M$, in particular $\mathbb{M} \succ M$, is also MS-measurable.*
- (3) *A definable set X is finite if and only if $\dim(X) = 0$ (in which case $\text{meas}(X) = |X|$). Moreover, for every formula $\varphi(x, y) \in L$ there is some $k_\varphi \in \mathbb{N}$ such that: for any $b \in M^y$, $\varphi(M, b)$ is finite if and only if $|\varphi(M, b)| \leq k_\varphi$ (follows from the finiteness of the set D_φ above).*
- (4) *Suppose that $\dim(M) = e$ and $\text{meas}(M) = \nu$. Then we can define a normalized measuring function by taking $\text{meas}'(X) = \frac{\text{meas}(X)}{\nu^{\dim(X)/e}}$ for each definable $X \subseteq M$. Then \dim, meas' also satisfy all the properties above, but in addition $\text{meas}'(M) = 1$ (and hence also $\text{meas}'(M^n) = 1$ for all $n \in \mathbb{N}$).*

- (5) For any definable (with parameters) set $B \subseteq \mathbb{M}^y$, we have a global Keisler measure μ_B (working in $\mathbb{M} \succ M$) with $\mu_B(B) = 1$ defined by

$$\mu_B(X) = \begin{cases} \frac{\text{meas}(X \cap B)}{\text{meas}(B)} & \text{if } \dim(X \cap B) = \dim(B), \\ 0 & \text{if } \dim(X \cap B) < \dim(B) \end{cases}$$

for any definable $X \subseteq M^y$.

From now on, for simplicity of exposition we will assume that $\dim(X) \leq n$ and that meas is normalized.

Fact 7.3. By [CVDDM92] (generalizing the classical Lang-Weil estimates using partial quantifier elimination of Ax), if $\varphi(x; y) \in L_{\text{ring}}$ with $|x| = n, |y| = m$ then there exist $C \in \mathbb{N}$ and a finite set H_φ of pairs $(d, \mu) \in \{0, 1, \dots, n\} \times \mathbb{Q}_{>0}$ so that for any finite field \mathbb{F}_q and $\bar{a} \in \mathbb{F}_q^m$,

$$\left| |\varphi(\mathbb{F}_q^n, \bar{a})| - \mu q^d \right| \leq Cq^{d-\frac{1}{2}}$$

for some $(d, \mu) \in H_\varphi$. Moreover, for each $(d, \mu) \in H_\varphi$ there is a formula $\psi_{d, \mu}(y) \in L_{\text{ring}}$ so that for all finite fields \mathbb{F}_q , $\psi_{d, \mu}(\mathbb{F}_q^m)$ defines the set of all tuples $\bar{a} \in \mathbb{F}_q^m$ satisfying this. This implies that any infinite ultraproduct of finite fields $F := \prod_{i \in \mathbb{N}} \mathbb{F}_{q_i} / \mathcal{U}$ is MS-measurable, where given $\varphi(x, y) \in L_{\text{ring}}$ and $\bar{a} \in F^m$, where $\bar{a} = (\bar{a}_i)_{i \in \mathbb{N}} / \mathcal{U}$ with $\bar{a}_i \in \mathbb{F}_{q_i}^m$, one sets $(\dim, \text{meas})(\varphi(F, \bar{a})) := (d, \mu)$ for (d, μ) as above (see [EM08, Proposition 3.9]).

Lemma 7.4. Let M be MS-measurable. For each definable $B \subseteq \mathbb{M}^x$, the Keisler measure μ_B is definable (over the same parameters as B), in a strong form (by Fact 7.2). And for any definable sets $B_i \subseteq M^{x_i}$ and permutation $\sigma : [k] \rightarrow [k]$, we have $\mu_{B_1 \times \dots \times B_k} = \mu_{B_1} \otimes \dots \otimes \mu_{B_k} = \mu_{B_{\sigma(1)}} \otimes \dots \otimes \mu_{B_{\sigma(k)}}$.

Proof. Consider $k = 2$. Let $\varphi(x_1, x_2) \in L(\mathbb{M})$ be arbitrary, and let $M \prec \mathbb{M}$ be a small model containing the parameters of B_1, B_2, φ . As both $\mu_{B_1} \otimes \mu_{B_2}$ and $\mu_{B_1 \times B_2}$ give measure one to $B_1 \times B_2$, we may replace $\varphi(x_1, x_2)$ by $\varphi(x_1, x_2) \wedge B_1(x_1) \wedge B_2(x_2)$ and assume $\varphi(x_1, x_2)$ defines a subset of $B_1 \times B_2$.

Let Y be the (M -definable) projection onto the second coordinate of the set defined by $\varphi(x_1, x_2)$. Then there are $r \in \mathbb{N}$ and $(d_i, m_i) \in (\mathbb{N} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ so that, taking $Y_i := \{b \in Y : (\dim, \text{meas})(\varphi(x_1, b)) = (d_i, m_i)\}$, $Y = Y_1 \sqcup \dots \sqcup Y_r$ is a partition of Y into non-empty disjoint M -definable sets. Note $Y_i \subseteq B_2$. Let $(\dim, \text{meas})(Y_i) = (e_i, \nu_i)$. Let $c := \max\{d_i + e_i : 1 \leq i \leq r\}$, and let $I := \{1 \leq i \leq r : d_i + e_i = c\}$. Then, by ‘‘Fubini’’ (see [MS08, Proposition 5.7]), $(\dim, \text{meas})(\varphi(x_1, x_2)) = (c, \sum_{i \in I} m_i \nu_i)$.

By definition of \otimes , additivity of Lebesgue integral and ‘‘Fubini’’ in a measurable structure we have (where $[Y_i] = \{p_2 \in S_{x_2}(M) : p_2 \vdash Y_i\}$):

$$\begin{aligned}
 \mu_{B_1} \otimes \mu_{B_2}(\varphi(x_1, x_2)) &= \int_{p_2 \in S_{x_2}(M)} \mu_{B_1}(\varphi(x_1, p_2)) d(\mu_{B_2})|_M(p_2) = \\
 \sum_{i=1}^r \int_{p_2 \in [Y_i]} \mu_{B_1}(\varphi(x_1, p_2)) d(\mu_{B_2})|_M(p_2) &= \sum_{\{i: d_i = \dim(B_1)\}} \frac{m_i}{\text{meas}(B_1)} \cdot \mu_{B_2}(Y_i) = \\
 \sum_{\{i: d_i = \dim(B_1) \wedge e_i = \dim(B_2)\}} \frac{m_i}{\text{meas}(B_1)} \cdot \frac{\nu_i}{\text{meas}(B_2)} &= \\
 \frac{\sum_{\{i: d_i = \dim(B_1) \wedge e_i = \dim(B_2)\}} m_i \cdot \nu_i}{\text{meas}(B_1 \times B_2)}. &
 \end{aligned}$$

Note that $d_i \leq \dim(B_1)$ and $e_i \leq \dim(B_2)$ for all $i \in [r]$. It follows that $\dim(\varphi(x_1, x_2)) = \dim(B_1 \times B_2) = \dim(B_1) + \dim(B_2)$ if and only if $c = \dim(B_1) + \dim(B_2)$, if and only if $I = \{i : d_i = \dim(B_1) \wedge e_i = \dim(B_2)\}$. Hence, by the calculation above, $\mu_{B_1 \times B_2}(\varphi(x_1, x_2)) = \mu_{B_1} \otimes \mu_{B_2}(\varphi(x_1, x_2))$.

Repeating the argument changing the order of integration and taking projection onto the first coordinate, we similarly get $\mu_{B_1 \times B_2}(x_1, x_2) = \mu_{B_2}(x_2) \otimes \mu_{B_1}(x_1)$, hence $\mu_{B_1} \otimes \mu_{B_2} = \mu_{B_2} \otimes \mu_{B_1}$.

The claim follows by induction for arbitrary k using that \otimes is associative on definable measures in arbitrary theories by Fact 3.4. \square

First we consider definable families of subsets of maximal dimension and measure bounded away from zero:

Lemma 7.5. *Let M be MS-measurable. For any partitioned formulas $\varphi(x, y)$, $\psi(x, z) \in L$ there is some $\alpha = \alpha(\varphi, \psi) > 0$ such that the following holds.*

Assume $e \in \mathbb{M}^z$ and $B \subseteq \mathbb{M}^y$ is definable (with parameters) so that, letting $X := \psi(\mathbb{M}, e)$, for all $b \in B$ we have: $\dim(\varphi(\mathbb{M}, b) \cap X) = \dim(X)$ and $\varphi(\mathbb{M}, b) \cap X$ is non-empty. Then there is some $a \in X$ such that

$$\dim(\varphi(a, \mathbb{M}) \cap B) = \dim(B) \text{ and } \text{meas}(\varphi(a, \mathbb{M}) \cap B) \geq \alpha \cdot \text{meas}(B).$$

Proof. Let $\varphi(x, y), \psi(x, z)$ be given. Let $\theta(x; y, z) := \varphi(x, y) \wedge \psi(x, z)$, and let $n \in \mathbb{N}$ and a finite set $D_\theta = \{(d_i, \mu_i) : i \in [n]\}$ be as given by Fact 7.2(1) for $\theta(x; y, z)$. Let $\alpha := \min\{\mu_i : i \in [n], \mu_i > 0\} > 0$ (depends only on φ, ψ). Then, for all $b \in B$, as $\varphi(\mathbb{M}, b) \cap X \neq \emptyset$ we have $\text{meas}(\varphi(\mathbb{M}, b) \cap X) \geq \alpha$ (by Definition 7.1(1),(2)).

Let $d := \dim(X)$. Consider the definable set

$$Z := \{(a, b) : a \in X, b \in B, \models \varphi(a, b)\} \subseteq X \times B.$$

Let $\pi_y : Z \rightarrow B$ be the projection of Z onto the y -coordinate. Then π_y is a definable surjection from Z onto B (as for all $b \in B$, $\varphi(\mathbb{M}, b) \cap X \neq \emptyset$ by assumption), $\dim(\pi_y^{-1}(b)) = \dim(\varphi(\mathbb{M}, b) \cap X) = d$ by assumption, and $\text{meas}(\pi_y^{-1}(b)) = \text{meas}(\varphi(\mathbb{M}, b) \cap X) \geq \alpha$. It follows by ‘‘Fubini’’ that $\dim(Z) = \dim(B) + d$ and $\text{meas}(Z) \geq \alpha \cdot \text{meas}(B)$.

On the other hand, let $A := \pi_x(Z) = \{a \in X : \models \exists b \varphi(a, b)\}$, where $\pi_x : Z \rightarrow A$ is the projection onto the x -coordinate. Again, π_x is a definable surjection onto A , and for any $a \in A$, $\pi_x^{-1}(a) = B \cap \varphi(a, \mathbb{M})$. By MS-measurability, there is $m \in \mathbb{N}$, $(d_i, \mu_i) \in \mathbb{N} \times \mathbb{R}_{>0} \cup \{(0, 0)\}$ for $i \in [m]$, and a partition $A = A_1 \sqcup \dots \sqcup A_m$ where each

$$A_i = \{a \in A : (\dim, \text{meas})(\pi_x^{-1}(a)) = (d_i, \mu_i)\}$$

is a non-empty definable set. Let $c := \max \{\dim(A_i) + d_i : i \in [m]\}$. Then by ‘‘Fubini’’ we have $\dim(Z) = c$ and $\text{meas}(Z) = \sum_{i \in I} \mu_i \cdot \text{meas}(A_i)$, where $I := \{i \in [m] : \dim(A_i) + d_i = c\}$. By the previous paragraph we have $\dim(Z) = d + \dim(B)$, and also for all $i \in [m]$ we have $\dim(A_i) \leq \dim(X) = d$ (as $A \subseteq X$) and $d_i \leq \dim(B)$ (as $\pi_x^{-1}(a) \subseteq B$ for all $a \in A$). Hence

$$I = \{i \in [m] : \dim(A_i) = d \wedge d_i = \dim(B)\}.$$

Now assume towards contradiction that $\text{meas}(\varphi(a, \mathbb{M}) \cap B) < \alpha \cdot \text{meas}(B)$ for all $a \in A$ with $\dim(\varphi(a, M) \cap B) = \dim(B)$. In particular $\mu_i < \alpha \cdot \text{meas}(B)$ for all $i \in I$. Also, as $\text{meas}(\mathbb{M}^x) = 1$ and $(A_i : i \in I)$ are disjoint non-empty subsets of \mathbb{M}^x of the same dimension $\dim(A_i) = d$, it follows (see e.g. [EM08, Lemma 3.3]) that $\sum_{i \in I} \text{meas}(A_i) = \text{meas}(\bigsqcup_{i \in I} A_i) \leq 1$. Hence $\text{meas}(Z) = \sum_{i \in I} \mu_i \cdot \text{meas}(A_i) < \alpha \cdot \text{meas}(B) \cdot \sum_{i \in I} \text{meas}(A_i) \leq \alpha \cdot \text{meas}(B)$. But by the first paragraph we had $\text{meas}(Z) \geq \alpha \cdot \text{meas}(B)$ — a contradiction. \square

Theorem 7.6. *Let M be an MS-measurable structure. Then every partitioned formula $\varphi(x, y) \in L$ with $|x| \leq d$ satisfies FHP _{$d+1$} with respect to the class of definable measures $\mathfrak{M}_y := \{\mu_B(y) : B \subseteq \mathbb{M}^y \text{ definable with parameters}\}$ (see Definition 3.5). In particular, $\varphi(x, y)$ satisfies FHP _{$d+1$} .*

Proof. The result will follow from a more precise claim:

Claim 7.7. For any partitioned formulas $\varphi(x, y), \psi(x, z) \in L$ and $\alpha \in \mathbb{R}_{>0}, d \in \mathbb{N}$ there is some $\beta = \beta(\varphi, \psi, d, \alpha) > 0$ such that the following holds.

Assume $e \in \mathbb{M}^z$ and $B \subseteq \mathbb{M}^y$ is definable (with parameters) so that, letting $X := \psi(\mathbb{M}, e)$, $\dim(X) \leq d$ and $(\mu_B)_{y_1, \dots, y_{d+1}}^{\otimes(d+1)}(\exists x \in X \wedge_{i \in [d+1]} \varphi(x, y_i)) \geq \alpha$. Then there exists some $a \in X$ so that $\mu_B(\varphi(a, y)) \geq \beta$.

Proof of Claim 7.7. We will prove the claim by induction on $d \in \mathbb{N}$. Let $\varphi(x, y), \psi(x, z), d, \alpha, X = \psi(\mathbb{M}, e)$ with $\dim(X) = d$ and B be given.

Base case $d = 0$.

The only possibility is $\dim(X) = 0$, hence X is finite and $|X| \leq k$ for some $k = k(\psi) \in \mathbb{N}$ by Fact 7.2(3), write $X = \{a_1, \dots, a_{k'}\}$ for some $k' \leq k$. By assumption

$$\alpha \leq (\mu_B)_{y_1}(\exists x \in X \varphi(x, y_1)) \leq (\mu_B)_{y_1} \left(\bigvee_{t \in [k']} \varphi(a_t, y_1) \right),$$

hence $(\mu_B)_{y_1}(\varphi(a_t, y_1)) \geq \frac{\alpha}{k}$ for at least one $t \in [k']$ — and we can take $\beta := \frac{\alpha}{k}$.

Inductive step $d > 0$.

Assume $d \geq 1$, $\dim(X) \leq d$ and the claim holds for all $0 \leq d' < d$. Let $\alpha_d := \frac{\alpha}{2}$.

Case 1. $\mu_B(B_d) \geq \alpha_d$ for $B_d := \{b \in B : \dim(\varphi(x, b) \wedge X(x)) = d\}$.

As $0 < \alpha_d \leq \mu_B(B_d)$, we have in particular $\dim(B_d) = \dim(B)$ (by Fact 7.2(5)) and $\text{meas}(B_d) \geq \alpha_d \cdot \text{meas}(B)$. As $d > 0$ we have $\varphi(\mathbb{M}, b) \cap X \neq \emptyset$ for all $b \in B_d$.

By Lemma 7.5 there is some $\gamma = \gamma(\varphi, \psi) > 0$ and some $a \in X$ so that

$$\begin{aligned} \dim(\varphi(a, \mathbb{M}) \cap B_d) &= \dim(B_d) = \dim(B) \text{ and} \\ \text{meas}(\varphi(a, \mathbb{M}) \cap B_d) &\geq \gamma \cdot \text{meas}(B_d) \geq \alpha_d \cdot \gamma \cdot \text{meas}(B), \end{aligned}$$

hence $\mu_B(\varphi(a, y)) \geq \beta_d := \alpha_d \cdot \gamma > 0$, and $\beta_d = \beta_d(\varphi, \psi, \alpha)$.

Case 2. $\mu_B(B_d) < \alpha_d$.

By assumption $(\mu_B)^{\otimes(d+1)}(C) \geq \alpha$ for

$$C := \{(b_1, \dots, b_{d+1}) \in B^{d+1} : \exists x \in X \bigwedge_{i \in [d+1]} \varphi(x, b_i)\}.$$

And $(\mu_B)^{\otimes(d+1)} = \mu_{B^{d+1}}$ by Lemma 7.4.

Let $B'_d := \{(b_1, \dots, b_{d+1}) \in B^{d+1} : \dim(\varphi(\mathbb{M}, b_{d+1}) \cap X) = d\}$. As $(\mu_B)^{\otimes(d+1)}$ extends the product measure we have $(\mu_B)^{\otimes(d+1)}(B'_d) = \mu_B(B_d) < \alpha_d$, hence $(\mu_B)^{\otimes(d+1)}(C \setminus B'_d) \geq \alpha - \alpha_d = \frac{\alpha}{2}$. Then by Fubini there exists some $b^* \in B \setminus B_d$ so that $(\mu_B)^{\otimes d}(\{(b_1, \dots, b_d) \in B^d : (b_1, \dots, b_d, b^*) \in C \setminus B'_d\}) \geq \frac{\alpha}{2}$.

Let $\psi'(x; z, y) := \psi(x, z) \wedge \varphi(x, y)$, $X' := X \cap \varphi(\mathbb{M}, b^*) = \psi'(\mathbb{M}; e, b^*)$. Then $\dim(X') \leq d - 1$, and we have $(\mu_B)^{\otimes d}_{y_1, \dots, y_d}(\exists x \in X' \bigwedge_{i \in [d]} \varphi(x, y_i)) \geq \frac{\alpha}{2}$. By the inductive assumption there exists some $\beta_{<d} = \beta_{<d}(\varphi, \psi', d - 1, \frac{\alpha}{2}) > 0$ and some $a \in X' \subseteq X$ so that $\mu_B(\varphi(a, y)) \geq \beta_{<d}$. Then $\beta := \min\{\beta_{<d}, \beta_d\} > 0$ is chosen depending only on φ, ψ, d, α and satisfies the requirement. This concludes the proof of the claim. \square

Now the theorem follows: given $\varphi(x, y)$ with $|x| \leq d$ and $\alpha > 0$, we can choose $\beta = \beta(\varphi, \alpha) > 0$ satisfying FHP _{$d+1$} with respect to \mathfrak{M}_y by applying the claim to $\varphi(x, y), \psi(x_1, \dots, x_d; z) := (\bigwedge_{i \in [d]} x_i = x_i), d, \alpha$. As all finite sets are definable, \mathfrak{M}_y contains all finitely supported probability measures on \mathbb{M}^y , so the ‘‘in particular’’ part follows by Remark 3.6. \square

Remark 7.8. In fact, the proof of Theorem 7.6 shows that given any finite set $D \subseteq \mathbb{N} \times \mathbb{R}_{>0} \cup \{(0, 0)\}$ and $\alpha \in \mathbb{R}_{>0}$ there exists $\beta = \beta(D, \alpha) \in \mathbb{R}_{>0}$ satisfying the following. Let M be any MS-measurable L -structure and $\varphi(x, y) \in L$ is an L -formula with $|x| \leq d$ so that, letting $(\bigwedge_t \varphi)(x; y_1, \dots, y_t) := \bigwedge_{i \in [t]} \varphi(x, y_i)$, we have $D_{\bigwedge_t \varphi} \subseteq D$ for all $t \in [d]$ (where $D_{\bigwedge_t \varphi}$ is the set given by Fact 7.2(1) for the formula $(\bigwedge_t \varphi)(x; y_1, \dots, y_t)$ in the MS-measurable structure M). Then $\varphi(x, y)$ satisfies FHP($d + 1, \alpha, \beta$) with respect to the class of measures \mathfrak{M}_y in M (see Definition 3.5).

As a corollary of Theorem 7.6, we get that definable families of sets of bounded description complexity in large finite fields satisfy the fractional Helly property, in the following sense. Given an L -structure M , we say that a family \mathcal{F} of subsets of M^d is *definable* if there exist partitioned L -formulas $\varphi(x, y), \psi(y, z)$, with $|x| = d$ and y, z finite tuples of variables, and $e \in M^z$ so that $\mathcal{F} = \{\varphi(M, b) : b \in \psi(M, e)\}$. We say that the *description complexity* of a definable family \mathcal{F} is $\leq (D_1, D_2)$ if there exist some φ of length $\leq D_1$ and some ψ of length $\leq D_2$ satisfying this for some e (see e.g. [Tao15, Section 1.2] for a discussion and examples). We view fields as structures in the ring language $L_{\text{ring}} = (+, \times, 0, 1)$.

Corollary 7.9. *For every $(D_1, D_2) \in \mathbb{N}$ and $\alpha > 0$ there exist $\beta = \beta(D_1, \alpha) > 0$ and $N = N(D_1, D_2, \alpha) \in \mathbb{N}$ satisfying the following.*

Let $d \in \mathbb{N}$, F a finite field with $|F| \geq N$ and $\mathcal{F} \subseteq F^d$ a definable family of sets of description complexity $\leq (D_1, D_2)$ so that

$$\left| \left\{ I \subseteq \mathcal{F} : |I| = d + 1 \wedge \bigcap_{S \in I} S \neq \emptyset \right\} \right| \geq \alpha \binom{|F|}{d + 1}.$$

Then $\bigcap_{S \in J} S \neq \emptyset$ for some $J \subseteq \mathcal{F}$ with $|J| \geq \beta|\mathcal{F}|$.

Proof. Fix (D_1, D_2) and $\alpha > 0$. Let Δ be the set of all L_{ring} -formulas of length $\leq D_1$, then Δ is finite. For each $\varphi(x, y) \in L_{\text{ring}}$ and $t \in \mathbb{N}$, let finite set $H_{\wedge_t \varphi}$ be as given by Fact 7.3 for $\wedge_t \varphi(x; y_1, \dots, y_t)$, and let $D := \bigcup \{H_{\wedge_t \varphi} : \varphi(x, y) \in \Delta, t \in [D_1]\}$, then D is still finite. It follows by Fact 7.3 that if $F := \prod_{i \in \mathbb{N}} F_i / \mathcal{U}$ is an arbitrary infinite ultraproduct of finite fields, $\varphi(x, y) \in L_{\text{ring}}$ is any formula of length $\leq D_1$ and $t \in [D_1]$, then F is MS-measurable and $D_{\wedge_t \varphi} \subseteq D$ (where $D_{\wedge_t \varphi}$ is given by Fact 7.2(1) for the formula $(\wedge_t \varphi)(x; y_1, \dots, y_t)$ in F). It follows by Theorem 7.6 and Remark 7.8 that there is $\beta = \beta(D, \alpha) = \beta(D_1, \alpha) > 0$ so that $\varphi(x, y)$ satisfies FHP($d+1, \alpha, \beta$) with respect to \mathfrak{M}_y in F .

Now assume towards contradiction that no $N \in \mathbb{N}$ satisfies the conclusion of the corollary with respect to $\beta/2$. Then for every $i \in \mathbb{N}$ there exists a finite field F_i with $|F_i| \geq i$ and a definable family $\mathcal{F}_i \subseteq F_i^d$ of description complexity $\leq (D_1, D_2)$, witnessed by some $\varphi_i(x, y_i), \psi_i(y_i, z_i), e_i \in F_i^{|z_i|}$, so that taking $B_i := \psi_i(F_i, e_i)$ and $C_i := \{(b_1, \dots, b_{d+1}) \in B_i^{d+1} : F_i \models \exists x \bigwedge_{t \in [d+1]} \varphi_i(x, b_t)\}$, we have $|C_i| \geq \alpha |B_i^{d+1}|$, but $|\varphi_i(a, F_i) \cap B_i| < \frac{\beta}{2} |B_i|$ for all $a \in F_i^{|x|}$. As there are only finitely many formulas of length at most $\max\{D_1, D_2\}$, passing to a subsequence we may assume $\varphi_i(x, y_i) = \varphi(x, y), \psi_i(y_i, z_i) = \psi(y, z)$ for all $i \in \mathbb{N}$. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} and $F := \prod_{i \in \mathbb{N}} F_i / \mathcal{U}, B := \prod_{i \in \mathbb{N}} B_i / \mathcal{U}, C := \prod_{i \in \mathbb{N}} C_i / \mathcal{U}$. Then F is an infinite MS-measurable structure, and by the definition of (\dim, meas) from Fact 7.3 we have $\dim(C) \geq \dim(B^{d+1})$ (hence equal to it as $C \subseteq B^{d+1}$) and $\text{meas}(C) \geq \alpha \text{meas}(B^{d+1})$, so using Łoś's theorem, Fact 7.2(5) and Lemma 7.4, we have

$$(\mu_B)^{\otimes(d+1)} \left(\left\{ (b_1, \dots, b_{d+1}) \in B^{d+1} : F \models \exists x \bigwedge_{t \in [d+1]} \varphi_i(x, b_t) \right\} \right) \geq \alpha.$$

Then, by the choice of β in the first paragraph, there is some $a \in F^{|x|}, a = (a_i)_{i \in \mathbb{N}} / \mathcal{U}$ with $a_i \in F_i^{|x|}$, so that $\mu_B(\varphi(a, F)) \geq \beta$, i.e. $\dim(\varphi(a, F) \cap B) = \dim(B)$ and $\text{meas}(\varphi(a, F) \cap B) \geq \beta \cdot \text{meas}(B)$. Again, by the definition of (\dim, meas) in F from Fact 7.3 this implies that $|\varphi_i(a_i, F_i) \cap B_i| \geq \beta |B_i|$ for a \mathcal{U} -large set of $i \in \mathbb{N}$ — a contradiction. \square

Corollary 7.10. *Assume M is an MS-measurable structure. Then M^x with $|x| \leq d$ satisfies colorful FHP _{$d+1$} with respect to \mathfrak{M} .*

Proof. By Theorem 7.6 and Proposition 4.6 (noting that the measures in \mathfrak{M} are definable and pairwise commuting by Lemma 7.4). \square

Similarly to Corollary 7.9, this implies colorful fractional Helly number $d+1$ for definable families of subsets of \mathbb{F}_q^d of bounded description complexity.

8. ULTRAPRODUCTS OF THE p -ADICS HAVE FHP

In this section we prove an Ax-Kochen-Ershov style result for the FHP property in henselian valued fields (Theorem 8.5). We only consider the equi-characteristic 0 case here (sufficient for the intended application to ultraproducts of the p -adics, Corollary 8.13) and leave the obvious more general questions for the future (see Problem 8.15).

8.1. Quantifier elimination in henselian valued fields. We recall some basic facts about the RV language for valued fields (we use [Fle11] as a reference, see also [ACGZ22, Section 5.1]). Fix a valued field K , with valuation $v : K^\times \rightarrow \Gamma$, value group Γ , residue field k and valuation ring \mathcal{O} . Let RV be the quotient group $K^\times / (1 + \mathfrak{m})$ where $\mathfrak{m} = \{x \in K : v(x) > 0\}$ is the maximal ideal of \mathcal{O} . We have a short exact sequence $1 \rightarrow k^\times \rightarrow \text{RV} \xrightarrow{v_{\text{rv}}} \Gamma \rightarrow 0$, where v_{rv} is induced by the valuation map and, for $a \in \mathcal{O} \setminus \mathfrak{m}$, the embedding $k^\times \rightarrow \text{RV}$ sends $a + \mathfrak{m} \in k^\times$ to $a(1 + \mathfrak{m}) \in \text{RV}^\times$.

Definition 8.1. We consider the two-sorted structure $M = (K, \text{RV}, \text{rv})$ in the language L (fixed for the rest of the section) consisting of:

- the quotient map $\text{rv} : K \rightarrow \text{RV}$,
- on the sort K , the ring structure,
- on the sort RV, the structure $\cdot, 1$ of a multiplicative group, a symbol 0 , a symbol ∞ and a ternary relation \oplus .

The multiplicative group structure on RV is interpreted as the group structure induced from K^\times and $0 \cdot x = x \cdot 0 = 0$, $\infty = \text{rv}(0)$. The relation \oplus is interpreted as the partially defined addition inherited from K : $\oplus(a, b, c) \iff \exists x, y, z \in K (a = \text{rv}(x) \wedge b = \text{rv}(y) \wedge c = \text{rv}(z) \wedge x + y = z)$. We let RV_* denote the structure $(\text{RV}, 0, 1, \cdot, \infty, \oplus)$, and denote its language by L_{RV_*} .

Remark 8.2. (1) Let $\text{WD}(x, y)$ be the (definable in RV_*) set of pairs of elements in RV for which the sum is well-defined as

$$\forall z, z' (\oplus(x, y, z) \wedge \oplus(x, y, z') \Rightarrow z = z').$$

Given a pair of elements $x, y \in \text{RV}$ such that $\text{WD}(x, y)$ holds, $x + y$ denotes the unique element $z \in \text{RV}$ satisfying $\oplus(x, y, z)$.

- (2) For $a, b \in K$, we have $\text{WD}(\text{rv}(a), \text{rv}(b)) \iff v(a + b) = \min\{v(a), v(b)\}$, in which case $\text{rv}(a + b) = \text{rv}(a) + \text{rv}(b)$ (see [Fle11, Proposition 2.4]).
- (3) The relation $v_{\text{rv}}(x) \leq v_{\text{rv}}(y)$ on RV is definable in RV_* [Fle11, Proposition 2.8(1)]. Hence the multiplicative group $k^\times \cong \ker v_{\text{rv}}$ is definable in RV_* , thus the ordered abelian group Γ is interpretable in RV_* , so using \oplus the field k is also interpretable in RV_* .

We will use the following Ax-Kochen-Ershov style relative quantifier elimination result:

Fact 8.3. [Fle11, Propositions 4.3 and 5.1] *Let K be a henselian valued field with $\text{char}(k) = 0$ and $M = (K, \text{RV}, \text{rv})$.*

- (1) *Suppose $S \subseteq K$ is A -definable in M , for some set of parameters A in M . Then there are $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in K \cap \text{acl}(A)$ and an $\text{acl}(A)$ -definable subset $D \subseteq \text{RV}^n$ such that*

$$S = \{x \in K : (\text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n)) \in D\}.$$

- (2) *RV_* is fully stably embedded (i.e. the structure on RV_* induced from M , with parameters, is precisely the one described above).*

Moreover, this holds with arbitrary additional structure on RV_ (see the discussion before [Fle11, Propositions 4.3]).*

Remark 8.4. (1) By compactness we get that the stable embeddedness in Fact 8.3(2) is *uniform*. I.e. for every $\varphi(x, y) \in L$ with x a tuple of sort RV and y a

tuple of arbitrary sorts there is some $\psi(x, z) \in L_{RV}$ satisfying the following: if $M = (K, RV, rv)$ is a henselian valued field of equicharacteristic 0 and b is a tuple in M corresponding to y , then there is some c a tuple in $RV(M)$ corresponding to z so that for all tuples a in $RV(M)$ corresponding x we have

$$M \models \varphi(a, b) \Leftrightarrow RV_*(M) \models \psi(a; c).$$

- (2) Using (1), Fact 8.3(1) and compactness, we have the following additional uniformity in Fact 8.3(1). For every $\varphi(x; y) \in L$, with x of sort K , $|x| = 1$, and y any finite tuple of variables of any sorts, we can choose $\psi(x_1, \dots, x_n; z) \in L_{RV}$ depending only on φ , with $|x_i| = 1$ of sort RV and z a finite tuple of variables of sort RV , so that: if $M = (K, RV, rv)$ is a henselian valued field of equicharacteristic 0 and b is a tuple in M corresponding to y , then there is some c a tuple in $RV(M)$ corresponding to z and some $\alpha_1, \dots, \alpha_n \in K(M)$ so that for any $a \in K(M)$,

$$M \models \varphi(a, b) \Leftrightarrow RV_*(M) \models \psi(rv(a - \alpha_1), \dots, rv(a - \alpha_n); c).$$

8.2. Ax-Kochen-Ershov for FHP. In this section we prove the following theorem:

Theorem 8.5. *Let $M = (K, RV, rv)$ be an equi-characteristic 0 henselian valued field (viewed as an L -structure, see Section 8.1). Then M satisfies FHP if and only if both the residue field k and the (ordered) value group Γ satisfy FHP.*

In the rest of the section we prove this theorem (we do not try to optimize the bounds). Throughout, we let $M = (K, RV, rv)$ be an equi-characteristic 0 henselian valued field, viewed as an L -structure, and $\mathbb{M} \succ M$ a monster model. By \aleph_1 -saturation, we may always assume that \mathbb{M} admits a cross-section map $ac : K(\mathbb{M}) \rightarrow k^\times(\mathbb{M})$, so we can view \mathbb{M} also as a structure in the language L_{ac} with the angular component map added to the language.

The distinction of the cases below is inspired by the analysis in [Che14, Section 7.2], [CS19] and [ACGZ22]. The following two lemmas are easy to verify by the basic properties of valuations (see the proofs of [Che10, Lemma 6] and [Che10, Lemma 7], replacing $c_{-\infty}, c_{+\infty}$ by c_0, c_n and restricting to $1 \leq i < j \leq n - 1$).

Lemma 8.6. *There is a finite set of formulas $\Delta \subseteq L$ such that the following holds for any $n \geq 3$. Let $(b_i : 0 \leq i \leq n)$ be a Δ -indiscernible sequence of singletons in \mathbb{K} , and consider the function $(i, j) \mapsto v(b_j - b_i)$ for $1 \leq i < j \leq n - 1$ (so we ignore the first and the last elements of the sequence). Then one of the following cases occurs:*

- (1) *it is strictly increasing depending only on i (in which case we say that the sequence $(b_i : 1 \leq i \leq n - 1)$ is pseudo-convergent),*
- (2) *it is strictly decreasing depending only on j (so the sequence $(b_i : 1 \leq i \leq n - 1)$ taken in the reverse direction is pseudo-convergent),*
- (3) *it is constant (we refer to such a sequence $(b_i : 1 \leq i \leq n - 1)$ as a “fan”).*

Lemma 8.7. *For any $n \geq 3$, let $(b_i : 0 \leq i \leq n)$ be a pseudo-convergent sequence of singletons from \mathbb{K} . Then for any $d \in \mathbb{K}$ there is some $0 \leq j \leq n$ such that:*

- (1) *for all $0 \leq i < j$: $v(b_n - b_i) < v(d - b_n)$, which implies $v(d - b_i) = v(b_n - b_i)$ and $ac(d - b_i) = ac(b_n - b_i)$,*

- (2) for all $n - 1 \geq i > j$: $v(b_n - b_i) > v(d - b_n)$, which implies $v(d - b_i) = v(d - b_n)$ and $\text{ac}(d - b_i) = \text{ac}(d - b_n)$.

First we consider the key special case:

Lemma 8.8. *Assume RV_* has FHP. Then every partitioned formula $\psi(x; y, z) \in L$ of the form $\varphi(\text{rv}(x - y), z)$, with $\varphi(x', z)$ an L_{RV} -formula and x, y singleton variables of sort K and z an arbitrary finite tuple of variables of sort RV , has FHP.*

Proof. Let $k = k(\varphi) \in \mathbb{N}$ be sufficiently large, to be determined in the proof. Let $\alpha > 0$ be arbitrary, and we find $\beta = \beta(\varphi, \alpha, k) > 0$ such that the following holds. For any $n \in \mathbb{N}$, let $(b_i : i \in [n])$ with $b_i \in K(\mathbb{M})$ and $(c_i : i \in [n])$ with $c_i \in \text{RV}(\mathbb{M})^z$ be arbitrary. Let $\mathcal{F} := \{\psi(\mathbb{M}; b_i, c_i) : i \in [n]\}$. Let $\mathcal{C}_0 := \text{Cons}_k(\mathcal{F})$ (we are using the notation from Section 2). Assume that $|\mathcal{C}_0| \geq \alpha \binom{n}{k}$ (and without loss of generality n is arbitrarily large with respect to k and α), then there is some $I \subseteq [n]$ such that $|I| \geq \beta n$ and $\{\psi(\mathbb{M}, b_i c_i) : i \in I\}$ is consistent.

Let finite $\Delta \subseteq L$ be as given by Lemma 8.6. Let $k_1 \in \mathbb{N}$ be arbitrary. Assuming that $k = k(k_1, \Delta)$ is sufficiently large, by Ramsey's theorem every sequence of elements from $K(\mathbb{M})$ of length k contains a Δ -indiscernible subsequence of length k_1 . Let $\mathcal{C}_1 := \{S \in \text{Cons}_{k_1}(\mathcal{F}) : (b_i : i \in S) \text{ is } \Delta\text{-indiscernible}\}$, where indiscernibility is with respect to the order induced from the natural order on $[n]$. Then for every $S \in \mathcal{C}_0$ there is some $S' \subseteq S$ such that $S' \in \mathcal{C}_1$. On the other hand, each $S' \in \mathcal{C}_1$ can be contained in at most $\binom{n-k_1}{k-k_1}$ sets $S \in \mathcal{C}_0$. Hence

$$|\mathcal{C}_1| \geq \frac{\alpha \binom{n}{k}}{\binom{n-k_1}{k-k_1}} \geq \alpha_1 \binom{n}{k_1},$$

for some $\alpha_1 = \alpha_1(\alpha, k, k_1) > 0$.

Now let

$$\begin{aligned} \mathcal{D}_1 &:= \{S \in \mathcal{C}_1 : (b_i : i \in S^-) \text{ is pseudo-convergent}\}, \\ \mathcal{D}_2 &:= \{S \in \mathcal{C}_1 : \text{the reverse of } (b_i : i \in S^-) \text{ is pseudo-convergent}\}, \\ \text{and } \mathcal{D}_3 &:= \{S \in \mathcal{C}_1 : (b_i : i \in S^-) \text{ is a fan}\}, \end{aligned}$$

where S^- is the set obtained from S by removing the minimal and the maximal elements (with respect to the order induced from $[n]$). By Lemma 8.6, $\mathcal{C}_1 = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, hence $|\mathcal{D}_i| \geq \frac{\alpha_1}{3} \binom{n}{k_1}$ must hold for at least one $i \in \{1, 2, 3\}$.

Case 1. $|\mathcal{D}_1| \geq \frac{\alpha_1}{3} \binom{n}{k_1}$.

For each $S \in \mathcal{D}_1$, fix some $a_S \in K(\mathbb{M})$ such that $a_S \models \{\psi(x; b_i, c_i) : i \in S\}$. Let $j_S \in S^-$ be as given by Lemma 8.7 for a_S and the sequence $(b_i : i \in S^-)$. By pigeon-hole, for every $S \in \mathcal{D}_1$ at least one of the sets $L_S := \{i \in (S^-)^- : i < j_S\}$, $R_S := \{i \in (S^-)^- : j_S < i\}$ must have size $\geq k_2 := \lfloor \frac{(k_1-4)}{2} \rfloor$. Let

$$\begin{aligned} \mathcal{E}_1 &:= \{S \in \text{Cons}_{k_2}(\mathcal{F}) : \exists S' \in \mathcal{D}_1 (S \subseteq L_{S'})\}, \\ \mathcal{E}_2 &:= \{S \in \text{Cons}_{k_2}(\mathcal{F}) : \exists S' \in \mathcal{D}_1 (S \subseteq R_{S'})\}, \\ \text{and } \mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2. \end{aligned}$$

As every $S \in \mathcal{D}_1$ contains some $S' \in \mathcal{E}$, by double counting as above we get

$$|\mathcal{E}| \geq \frac{\frac{\alpha_1}{3} \binom{n}{k_1}}{\binom{n-k_2}{k_1-k_2}} \geq \alpha_2 \binom{n}{k_2}$$

for some $\alpha_2 = \alpha_2(\alpha_1, k_1) > 0$. Hence $|\mathcal{E}_i| \geq \frac{\alpha_2}{2} \binom{n}{k_2}$ for at least one $i \in \{1, 2\}$.

Case 1.1. $|\mathcal{E}_1| \geq \frac{\alpha_2}{2} \binom{n}{k_2}$.

Unwinding the definition, for every $S \in \mathcal{E}_1$ there is some $a_S \in K(\mathbb{M})$ such that $a_S \models \{\psi(x; b_i, c_i) : i \in S\}$ and, denoting by i_S the last element of S in the order induced from $[n]$, $v(b_{i_S} - b_i) < v(a_S - b_{i_S})$, and hence $\text{rv}(a_S - b_i) = \text{rv}(b_{i_S} - b_i)$, for all $i \in S, i < i_S$ (see Lemma 8.7).

As there are at most n choices for i_S as S varies over \mathcal{E}_1 , it follows that for some $i^* \in [n]$, the set

$$\mathcal{G}_1 := \{S \in \text{Cons}_{k_2-1}(\mathcal{F}) : \exists S' \in \mathcal{E}_1 (S = S' \setminus \{i_{S'}\} \wedge i_{S'} = i^*)\}$$

has size at least $\frac{|\mathcal{E}_1|}{n} \geq \alpha_3 \binom{n}{k_2-1}$ for some $\alpha_3 = \alpha_3(\alpha_2, k_2) > 0$. Note that, assuming $k_2 - 1 \geq 1$, for $\mathcal{H} := \bigcup \mathcal{G}_1$ we have $|\mathcal{H}| \geq \left(\alpha_3 \binom{n}{k_2-1}\right)^{\frac{1}{k_2-1}} \geq \alpha_4 n$ for some $\alpha_4 = \alpha_4(\alpha_3, k_2) > 0$. Note that for every $i \in \mathcal{H}$ there is some $a_S \in K(\mathbb{M})$ such that $\models \psi(a_S; b_i, c_i)$, that is $\models \varphi(\text{rv}(a_S - b_i), c_i)$, and $\text{rv}(a_S - b_i) = \text{rv}(b_{i^*} - b_i)$. Combining, we get that $b_{i^*} \models \{\psi(x; b_i, c_i) : i \in \mathcal{H}\}$. Hence we have found a consistent subfamily \mathcal{F}_0 of \mathcal{F} of size $\geq \beta_1 n$ for $\beta_1 := \alpha_4 > 0$.

Case 1.2. $|\mathcal{E}_2| \geq \frac{\alpha_2}{2} \binom{n}{k_2}$.

Unwinding the definition, for every $S \in \mathcal{E}_2$ there is some $a_S \in K(\mathbb{M})$ such that $a_S \models \{\psi(x; b_i, c_i) : i \in S\}$ and, denoting by i_S the last element of S in the order induced from $[n]$, $v(a_S - b_i) < v(b_{i_S} - b_i)$, and hence $\text{rv}(a_S - b_i) = \text{rv}(a_S - b_{i_S})$, for all $i \in S, i < i_S$ (see Lemma 8.7).

Again, as there are at most n choices for i_S as S varies over \mathcal{E}_2 , it follows that for some $i^* \in [n]$, the set

$$\mathcal{G}_2 := \{S \in \text{Cons}_{k_2-1}(\mathcal{F}) : \exists S' \in \mathcal{E}_2 (S = S' \setminus \{i_{S'}\} \wedge i_{S'} = i^*)\}$$

has size at least $\frac{|\mathcal{E}_2|}{n} \geq \alpha_3 \binom{n}{k_2-1}$ for some $\alpha_3 = \alpha_3(\alpha_2, k_2) > 0$.

Let $\psi'(x'; z, z')$ be the L_{RV} -formula $\varphi(x', z) \wedge (v_{\text{rv}}(x') < v_{\text{rv}}(z'))$. Consider the family

$$\mathcal{F}' := \{\varphi(x', c_i) \wedge (v_{\text{rv}}(x') < v_{\text{rv}}(b_{i^*} - b_i)) : i \in [n]\} = \{\psi'(\mathbb{M}; c_i, d_i) : i \in [n]\}$$

of subsets of $\text{RV}(\mathbb{M})$, where $d_i := \text{rv}(b_{i^*} - b_i)$. Note that $\mathcal{G}_2 \subseteq \text{Cons}_{k_2-1}(\mathcal{F}')$, hence $|\text{Cons}_{k_2-1}(\mathcal{F}')| \geq \alpha_3 \binom{n}{k_2-1}$.

As $\psi'(x'; z, z')$ is an L_{RV} -formula, by assumption (and Remark 8.4(1)) it has FHP. Assuming $k_2 - 1$ is larger than the fractional Helly number of ψ' , there is some $\beta_2 = \beta_2(\psi', \alpha_3) > 0$ and $I \subseteq [n]$ such that $|I| \geq \beta_2 n$ and

$$\{\varphi(x', c_i) \wedge (v_{\text{rv}}(x') < v(b_{i^*} - b_i)) : i \in I\}$$

is consistent, say realized by $e \in \text{RV}(\mathbb{M})$ (where $v(b_{i^*} - b_i)$ is a shortcut for $v_{\text{rv}}(\text{rv}(b_{i^*} - b_i))$ in our language). Let $a \in K(\mathbb{M})$ be such that $\text{rv}(a - b_{i^*}) = e$. Then $v(a - b_{i^*}) = v_{\text{RV}}(e) < v(b_{i^*} - b_i)$ for all $i \in I$, hence $\text{rv}(a - b_i) = \text{rv}(a - b_{i^*}) = e$ for all $i \in I$. And so $a \models \{\varphi(\text{rv}(x - b_i), c_i) : i \in I\}$. Hence we have found a consistent subfamily \mathcal{F}_0 of \mathcal{F} of size $\geq \beta_2 n$.

Case 2. $|\mathcal{D}_2| \geq \frac{\alpha_1}{3} \binom{n}{k_1}$.

Reduces to Case 1 by working with the reverse order on $[n]$.

Case 3. $|\mathcal{D}_3| \geq \frac{\alpha_1}{3} \binom{n}{k_1}$.

For each $S \in \mathcal{D}_3$, the Δ -indiscernible sequence $(b_i : i \in S^-)$ is a fan, that is there is some $\gamma_S \in \Gamma$ such that $v(b_j - b_i) = \gamma_S$ for all $i < j \in S^-$. We fix some $a_S \in K(\mathbb{M})$ such that $a_S \models \{\psi(x; b_i, c_i) : i \in S\}$.

For each $S \in \mathcal{D}_3$, let

$$S_{<} := \{i \in S^- : v(a_S - b_i) < \gamma_S\}, S_{>} := \{i \in S^- : v(a_S - b_i) > \gamma_S\}$$

$$S_{=} := \{i \in S^- : v(a_S - b_i) = \gamma_S\}.$$

Then S^- is a disjoint union of $S_{<}, S_{>}, S_{=}$. Let $k_2 := \lfloor \frac{k_1-2}{3} \rfloor$. Let

$$\mathcal{K}_1 := \{S' \in \text{Cons}_{k_2}(\mathcal{F}) : \exists S \in \mathcal{D}_3 (S' \subseteq S_{<})\},$$

$$\mathcal{K}_2 := \{S' \in \text{Cons}_{k_2}(\mathcal{F}) : \exists S \in \mathcal{D}_3 (S' \subseteq S_{>})\},$$

$$\mathcal{K}_3 := \{S' \in \text{Cons}_{k_3}(\mathcal{F}) : \exists S \in \mathcal{D}_3 (S' \subseteq S_{=})\},$$

$$\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3.$$

As every $S \in \mathcal{D}_3$ contains some $S' \in \mathcal{K}$, by double counting we get

$$|\mathcal{K}| \geq \frac{\frac{\alpha_1}{3} \binom{n}{k_1}}{\binom{n-k_2}{k_1-k_2}} \geq \alpha_2 \binom{n}{k_2}$$

for some $\alpha_2 = \alpha_2(\alpha_1, k_1) > 0$. Hence $|\mathcal{K}_i| \geq \frac{\alpha_2}{3} \binom{n}{k_2}$ for some $i \in \{1, 2, 3\}$.

Case 3.1. $|\mathcal{K}_1| \geq \frac{\alpha_2}{3} \binom{n}{k_2}$.

If $S \in \mathcal{K}_1$, then in particular $v(a_S - b_i) < \gamma_S = v(b_{i_S} - b_i)$ for all $i \in S$, where i_S is the last element of S in the order induced from $[n]$. Hence $\text{rv}(a_S - b_i) = \text{rv}(a_S - b_{i_S})$ for all $i \in S, i < i_S$. Then we can conclude exactly as in Case 1.2.

Case 3.2. $|\mathcal{K}_2| \geq \frac{\alpha_2}{3} \binom{n}{k_2}$.

Note that if $S \in \mathcal{K}_2$, then $|S| \leq 1$. Indeed, if $v(a_S - b_i) > \gamma_S = v(b_i - b_j)$ then $v(b_i - b_j) = v(a_S - b_j) > \gamma_S$, a contradiction. Hence, assuming $k_2 \geq 2$, this case does not occur.

Case 3.3. $|\mathcal{K}_3| \geq \frac{\alpha_2}{3} \binom{n}{k_2}$.

In this case for every $S \in \mathcal{K}_3$ we have $v(a_S - b_i) = v(b_j - b_i) = \gamma_S$ for all $i < j \in S$. Let i_S denote the last element of S in the order induced from $[n]$. It follows that $\text{rv}(a_S - b_i) = \text{rv}(a_S - b_{i_S}) + \text{rv}(b_{i_S} - b_i)$ for all $i \in S, i < i_S$.

There are at most n choices for i_S when S varies over \mathcal{K}_3 . It follows that for some $i^* \in [n]$, the set

$$\mathcal{L} := \{S \in \text{Cons}_{k_2-1}(\mathcal{F}) : \exists S' \in \mathcal{K}_3 (S = S' \setminus \{i_{S'}\} \wedge i_{S'} = i^*)\}$$

has size at least $\frac{|\mathcal{K}_3|}{n} \geq \alpha_3 \binom{n}{k_2-1}$ for some $\alpha_3 = \alpha_3(\alpha_2, k_2) > 0$.

Let $\psi'(x'; z, z')$ be the L_{RV} -formula

$$\varphi(x' + z', z) \wedge (v_{\text{rv}}(x') = v_{\text{rv}}(z')) \wedge \text{WD}(x', z')$$

(see Remark 8.2), and let \mathcal{F}' be the family

$$\{\varphi(x' + \text{rv}(b_{i^*} - b_i), c_i) \wedge (v_{\text{rv}}(x') = v(b_{i^*} - b_i)) \wedge \text{WD}(x', \text{rv}(b_{i^*} - b_i)) : i \in [n]\}$$

$$= \{\psi'(x'; c_i, d_i) : i \in [n]\}$$

of definable subsets of $\text{RV}(\mathbb{M})$, with $d_i := \text{rv}(b_{i^*} - b_i)$. Note that $\mathcal{L} \subseteq \text{Cons}_{k_2-1}(\mathcal{F}')$, hence $|\text{Cons}_{k_2-1}(\mathcal{F}')| \geq \alpha_3 \binom{n}{k_2-1}$.

As $\psi'(x'; z, z')$ is an L_{RV} -formula, by assumption it has FHP. If $k_2 - 1$ is larger than the fractional Helly number of ψ' , then there is some $\beta_3 = \beta_3(\psi', \alpha_3) > 0$ and $I \subseteq [n]$ such that $|I| \geq \beta_3 n$ and $\{\psi'(x'; c_i, d_i) : i \in I\}$ is consistent, realized by some $e \in \text{RV}(\mathbb{M})$.

Let $a \in K(\mathbb{M})$ be such that $\text{rv}(a - b_{i^*}) = e$. Then in particular $v(b_{i^*} - b_i) = v_{\text{rv}}(e) = v(a - b_{i^*})$ for all $i \in I$ and, as $\text{WD}(e, \text{rv}(b_{i^*} - b_i))$ holds, we have $\text{rv}(a - b_i) = \text{rv}(a - b_{i^*}) + \text{rv}(b_{i^*} - b_i) = e + \text{rv}(b_{i^*} - b_i)$ for all $i \in I$. Hence $a \models \{\varphi(\text{rv}(x - b_i), c_i) : i \in I\}$, and we have found a consistent subfamily \mathcal{F}_0 of \mathcal{F} of size $\geq \beta_3 n$.

Finally, choosing k sufficiently large — satisfying all of the assumptions in each of the cases, which can be done depending only on the formula φ — and taking $\beta := \min\{\beta_i\}$ with β_i appearing in each of the cases, we have demonstrated the claim. \square

Proposition 8.9. *Assume RV_* satisfies FHP. Then every partitioned formula $\varphi(x, y) \in L$ satisfies FHP.*

Proof. By Fact 2.11(3) it suffices to show FHP for every partitioned formula $\varphi(x, y) \in L$ with $|x| = 1$. If x is a singleton of sort RV , this holds by Remark 8.4(1) and the assumption that RV_* satisfies FHP. So assume that x is of sort K . By Fact 8.3 and Remark 8.4(2), it suffices to show that FHP holds for every partitioned formula of the form

$$\psi(x; y_0, \dots, y_{m-1}, z) = \varphi(\text{rv}(x - y_0), \dots, \text{rv}(x - y_{m-1}), z),$$

where x is a singleton of sort K and $\varphi(x_0, \dots, x_{m-1}, z)$ is an L_{RV} -formula (with x_i singletons and z an arbitrary finite tuple). We prove this by induction on m , the base case $m = 1$ given by Lemma 8.8.

Let $k = k(\varphi)$ be sufficiently large (to be determined in the proof) and let $\alpha > 0$ be arbitrary. Let $n \in \mathbb{N}$, $(b_{i,t} : i \in [n], t < m)$ with $b_{i,t} \in K(\mathbb{M})$ and $(c_i : i \in [n])$ with $c_i \in \text{RV}(\mathbb{M})^z$ be arbitrary (without loss of generality n is arbitrarily large with respect to k and α), and let $\mathcal{F} := \{\psi(x; b_{i,0}, \dots, b_{i,m-1}, c_i) : i < n\}$ be such that $|\text{Cons}_k(\mathcal{F})| \geq \alpha \binom{n}{k}$. Let $\mathcal{C} := \text{Cons}_k(\mathcal{F})$. For each $S \in \mathcal{C}$, fix some $a_S \in K(\mathbb{M})$ realizing $\{\psi(x; b_i, c_i) : i \in S\}$, where $b_i := (b_{i,0}, \dots, b_{i,m-1})$. Let

$$\begin{aligned} S_1 &:= \{i \in S : v(a_S - b_{i,0}) < v(b_{i,m-1} - b_{i,0})\} = \\ &\quad \{i \in S : v(a_S - b_{i,m-1}) < v(b_{i,m-1} - b_{i,0})\}, \\ S_2 &:= \{i \in S : v(a_S - b_{i,0}) > v(b_{i,m-1} - b_{i,0})\}, \\ S_3 &:= \{i \in S : v(a_S - b_{i,m-1}) > v(b_{i,m-1} - b_{i,0})\} \text{ and} \\ S_4 &:= \{i \in S : v(a_S - b_{i,0}) = v(a_S - b_{i,m-1}) = v(b_{i,m-1} - b_{i,0})\}. \end{aligned}$$

Then S is the disjoint union of S_1, S_2, S_3, S_4 . Let $k_2 := \lfloor \frac{k}{4} \rfloor$,

$$\mathcal{D}_i := \{S' \in \text{Cons}_{k_2}(\mathcal{F}) : \exists S \in \mathcal{C} (S' \subseteq S_i)\}$$

for $i \in \{1, 2, 3, 4\}$, and $\mathcal{D} := \bigcup_{i=1}^4 \mathcal{D}_i$. As every $S \in \mathcal{C}$ contains some $S' \in \mathcal{D}$, by double counting we get

$$|\mathcal{D}| \geq \frac{\alpha \binom{n}{k}}{\binom{n-k_2}{k-k_2}} \geq \alpha_2 \binom{n}{k_2}$$

for some $\alpha_2 = \alpha_2(\alpha, k) > 0$. Hence $|\mathcal{D}_i| \geq \frac{\alpha_2}{4} \binom{n}{k_2}$ for some $i \in \{1, 2, 3, 4\}$.

Case 1. $|\mathcal{D}_1| \geq \frac{\alpha_2}{4} \binom{n}{k_2}$.

For every $S \in \mathcal{D}_1$, we have $v(a_S - b_{i,0}) < v(b_{i,m-1} - b_{i,0})$, which implies $\text{rv}(a_S - b_{i,0}) = \text{rv}(a_S - b_{i,m-1})$, for all $i \in S$. We let

$$\begin{aligned} \psi'(x; y_0, \dots, y_{m-2}, z, z') &:= \\ \varphi(\text{rv}(x - y_0), \dots, \text{rv}(x - y_{m-2}), \text{rv}(x - y_0), z) &\wedge (v(x - y_0) < v_{\text{rv}}(z')). \end{aligned}$$

Case 2. $|\mathcal{D}_2| \geq \frac{\alpha_2}{4} \binom{n}{k_2}$.

For every $S \in \mathcal{D}_2$ we have $v(a_S - b_{i,0}) > v(b_{i,m-1} - b_{i,0})$, so $\text{rv}(a_S - b_{i,m-1}) = \text{rv}(b_{i,m-1} - b_{i,0})$, for all $i \in S$. We define

$$\begin{aligned} \psi'(x; y_0, \dots, y_{m-2}, z, z') &:= \\ \varphi(\text{rv}(x - y_0), \dots, \text{rv}(x - y_{m-2}), z', z) &\wedge (v(x - y_0) > v_{\text{rv}}(z')). \end{aligned}$$

Case 3. $|\mathcal{D}_3| \geq \frac{\alpha_2}{4} \binom{n}{k_2}$.

Symmetric to Case 2. We define

$$\begin{aligned} \psi'(x; y_1, \dots, y_{m-1}, z, z') &:= \\ \varphi(z', \text{rv}(x - y_1), \dots, \text{rv}(x - y_{m-1}), z) &\wedge (v(x - y_{m-1}) > v_{\text{rv}}(z')). \end{aligned}$$

Case 4. $|\mathcal{D}_4| \geq \frac{\alpha_2}{4} \binom{n}{k_2}$.

For every $S \in \mathcal{D}_4$ and $i \in S$ we have

$$\begin{aligned} v(a_S - b_{i,0}) &= v(a_S - b_{i,m-1}) = v(b_{i,m-1} - b_{i,0}), \text{ hence} \\ \text{rv}(a_S - b_{i,0}) &= \text{rv}(a_S - b_{i,m-1}) + \text{rv}(b_{i,m-1} - b_{i,0}). \end{aligned}$$

We define (see Remark 8.2)

$$\begin{aligned} \psi'(x; y_1, \dots, y_{m-1}, z, z') &:= \\ \varphi(\text{rv}(x - y_{m-1}) + z', \text{rv}(x - y_1), \dots, \text{rv}(x - y_{m-1}), z) &\wedge \text{WD}(\text{rv}(x - y_{m-1}), z'). \end{aligned}$$

We let $d_i := \text{rv}(b_{i,m-1} - b_{i,0})$ for $i \in [n]$. In each of the cases, we consider the family $\mathcal{F}' := \{\psi'(x; b_i, c_i, d_i) : i \in [n]\}$. By the inductive assumption for $m-1$, the formula ψ' has FHP. Assuming that k_2 is larger than the fractional Helly number of ψ' , it follows that there is some $\beta = \beta(\frac{\alpha_2}{4}, \psi', k_2) > 0$ and $I \subseteq [n]$ such that $|I| \geq \beta n$ and $\{\psi'(x; b_i, c_i, d_i) : i \in I\}$ is consistent, realized by some $a \in K(\mathbb{M})$. In each of the four cases, it is easy to see that then also $a \models \{\psi(x; b_{i,0}, \dots, b_{i,m-1}, c_i) : i \in I\}$. For example, this holds in Case 4 as by the choice of a we have that $\text{WD}(\text{rv}(a - b_{i,m-1}), \text{rv}(b_{i,m-1} - b_{i,0}))$ holds for all $i \in I$, and so $\text{rv}(a - b_{i,0}) = \text{rv}(a - b_{i,m-1}) + \text{rv}(b_{i,m-1} - b_{i,0})$ for all $i \in I$.

Finally, choosing k sufficiently large — satisfying all of the assumptions in each of the cases, which can be done depending only on the formula φ — and taking the minimal β appearing in each of the cases, we have demonstrated the claim. \square

Proposition 8.10. *Assume k and Γ satisfy FHP. Then RV_* also satisfies FHP.*

Proof. Working in \mathbb{M} , which is in particular \aleph_1 -saturated, we have that the short exact sequence $1 \rightarrow k^\times \rightarrow \text{RV} \xrightarrow{\text{vry}} \Gamma \rightarrow 0$ splits (see e.g. [AvdHvdD17, Corollary 3.3.38]), so RV is the direct sum of k^* and Γ . We have that RV_* is bi-interpretable with this short exact sequence (see [ACGZ22, Lemma 5.17]), and this short exact sequence expanded by a function symbol for a right inverse map for vry is bi-interpretable with a structure consisting of two disjoint sorts given by k and Γ with their induced structure (and no additional structure). It is then clear, using Fact 2.11(3), that if both k and Γ are FHP, then such a structure is also FHP. By Lemma 2.11, any theory interpretable in an FHP theory is also FHP, so we conclude that RV_* is FHP. \square

Combining Proposition 8.10 and Proposition 8.9 finishes the proof of Theorem 8.5.

8.3. FHP in ultraproducts of \mathbb{Q}_p and explicit bounds. Burden in henselian valued fields is studied extensively [Che10, Che14, CH14, CS19, Sin22, Tou23]. In particular we have:

Fact 8.11. [CS19] *If $M = (K, \text{RV}, \text{rv})$ is a henselian valued field of equicharacteristic 0, $\text{bdn}(k) = \text{bdn}(\Gamma) \leq 1$ and $k^\times / (k^\times)^p$ is finite for all prime p , then $\text{bdn}(M) = 1$. In particular, $\text{bdn}(M) = 1$ for any ultraproduct $K := \prod_{i \in \mathbb{N}} \mathbb{Q}_{p_i} / \mathcal{U}$ with p_i prime.*

Remark 8.12. If k is inp-minimal, then, by the proof of [CKS15, Corollary 4.6], $k^\times / (k^\times)^p$ is finite for all but at most one prime p . It is open if it has to be finite for all p [CS19, Problem 25] (true when k is also NIP by [Joh18]).

Corollary 8.13. *Let K be $\prod_{i \in \mathbb{N}} \mathbb{Q}_{p_i} / \mathcal{U}$ or $\prod_{i \in \mathbb{N}} \mathbb{F}_{p_i}((t)) / \mathcal{U}$ for some prime p_i and a non-principal ultrafilter \mathcal{U} on \mathbb{N} . Let $M = (K, \text{RV}, \text{rv})$. Then every partitioned formula $\varphi(x, y) \in L$ with $|x| \leq d$ satisfies FHP_{2^d} .*

Proof. The residue field $k = \prod_{i \in \mathbb{N}} \mathbb{F}_{p_i} / \mathcal{U}$ is pseudo-finite, hence satisfies FHP by Theorem 7.6 (and Fact 7.3). The (ordered abelian) value group Γ is NIP by [GS84], hence satisfies FHP by Fact 2.23. Hence $M = (K, \text{RV}, \text{rv})$ satisfies FHP by Theorem 8.5.

By Fact 8.11 we have $\text{bdn}(M) = 1$. By Corollary 4.10 we know that, assuming M satisfies FHP, any formula $\varphi(x, y) \in L$ satisfies $\text{FHP}_{\text{bdn}(M^x)+1}$. By Fact 4.5 this implies that any formula $\varphi(x, y) \in L$ satisfies $\text{FHP}_{(\text{bdn}(M)+1)^{|x|}}$ (and if M is NIP, satisfies $\text{FHP}_{|x| \cdot \text{bdn}(M)+1}$). Hence any formula $\varphi(x, y) \in L$ satisfies $\text{FHP}_{2^{|x|}}$. \square

We note that each individual field $\mathbb{F}_p((t))$ has TP_2 by [CKS15, Corollary 3.3], hence cannot satisfy FHP.

Conjecture 8.14. *The bound 2^d in Corollary 8.13 can be improved to $d + 1$.*

- Problem 8.15.**
- (1) Which (ordered, valued, etc.) fields satisfy FHP?
 - (2) Do bounded PAC, pRC, pPC fields satisfy FHP? They are known to be NTP_2 (see [MRK25]).
 - (3) Does VFA_0 satisfy FHP? Again, it is known to be NTP_2 by [CH14]. In particular, does ACFA satisfy FHP?

9. ULCFS AND COUNTING PARTIAL TYPES OVER FINITE SETS

9.1. ULCFS. Let T be a complete first-order theory with infinite models in a language L . First we consider finitary counterparts of some notions, properties and facts around forking and dividing. We assume some familiarity with the basic definitions and properties of dividing (see e.g. [CK12]).

Definition 9.1. Let Δ be a set of formulas and $n, k \in \mathbb{N}$. Given a formula $\varphi(x, b)$ with $\varphi(x, y) \in L$ and b a tuple in \mathbb{M} , we say that it (Δ, n, k) -divides over a set $C \subseteq \mathbb{M}$ if there is a $\Delta(C)$ -indiscernible sequence $(b_i : i < n)$ in \mathbb{M} so that $b_0 = b$ and the set of formulas $\{\varphi(x, b_i) : i < n\}$ is k -inconsistent.

Note that, by compactness, $\varphi(x, b)$ is k -dividing over C if and only if it (Δ, n, k) -dividing over C for all finite sets of formulas $\Delta \subseteq L$ and all $n \in \omega$.

Definition 9.2. Let $\pi(x)$ be a partial type over a set of parameters B , and let $C \subseteq B$.

- (1) For a formula $\varphi(x, y) \in L$, we say that π is (φ, Δ, n, k) -dividing over C if there is an instance $\varphi(x, b) \in \pi$ such that $\varphi(x, b)$ is (Δ, n, k) -dividing over C .
- (2) We say that π internally (φ, Δ, n, k) -divides over C if it (φ, Δ, n, k) -divides over C , and moreover we can choose a $\Delta(C)$ -indiscernible sequence $(b_i : i < n)$ witnessing (Δ, n, k) -dividing over C as in Definition 9.1 inside $B = \text{dom}(\pi)$.

Remark 9.3. Assume that $p \in S_x(M)$ is a type over a model M and C is a subset of M . If M is $|C|^+$ -saturated, then dividing over C is equivalent to internal dividing over C for p (given an arbitrary sequence witnessing dividing, by saturation we can find a copy of it inside M). For example, if $C \subseteq M$ is finite, then for any finite Δ, n, k we have $p \in S(M)$ is (Δ, n, k) -dividing over C if and only if it internally (Δ, n, k) -dividing over C . We will be mostly interested in types over finite sets here, where these two notions may be different.

Definition 9.4. (1) We say that dividing in T satisfies the *Uniform Local Character over Finite Sets* (or that T satisfies ULCFS) if every partitioned formula $\varphi(x, y) \in L$ satisfies ULCFS, i.e. for every $k \in \omega$ there is some finite set of formulas $\Delta \subseteq L$ and some $n \in \omega$ so that: for every finite set B and every $p(x) \in S_\varphi(B)$ there is some $C \subseteq B$ with $|C| \leq n$ such that p does not internally (φ, Δ, n, k) -divide over C .

(2) We say that T satisfies *strong ULCFS* if in (1) instead of internal non-dividing we require non-dividing.

Obviously, strong ULCFS implies ULCFS.

Problem 9.5. Is it true that (strong) ULCFS holds in T only assuming that Definition 9.4 is satisfied for $k = 2$? Only assuming that Definition 9.4 holds for all formulas $\varphi(x, y)$ with $|x| = 1$?

9.2. NIP implies strong ULCFS. The following is the so-called *UDTFS* property (i.e. *Uniform Definability of Types over Finite Sets*). It was conjectured by Laskowski, and, after some partial results [JL10, Gui12], established in [CS13, CS15] for formulas in NIP theories (and more recently for NIP formulas in arbitrary theories [EK21]):

Fact 9.6. *Let $\varphi(x, y) \in \mathcal{L}$ be a partitioned NIP formula. Then there exists a formula $\theta(y; \bar{y}) \in \mathcal{L}$, with $\bar{y} = (y_1, \dots, y_d)$ for some $d \in \mathbb{N}$, satisfying the following: for every finite $A \subseteq \mathbb{M}^y$ with $|A| \geq 2$ and every $b \in \mathbb{M}^x$, there exists some tuple $c \in A^d$ such that for every $a \in A$ we have $\models \varphi(b, a) \iff \models \theta(a, c)$.*

It is immediate that if $\varphi(x, y)$ satisfies UDTFS then it also satisfies ULCFS. Indeed, given $\theta(y, \bar{y})$ as in Fact 9.6 and finite set A , every $p(x) \in S_\varphi(A)$ clearly does not internally (φ, θ, n, k) -divide over a subset of A size at most d for any $n, k \in \mathbb{N}$, namely over c where c is so that $\theta(y, c)$ is a definition for p ($\varphi(x, a) \in p \Rightarrow \models \theta(a, c)$), so if $a_i \in A$ for $i < n$ with $a = a_0$ and $(a_i : i < n)$ is θ -indiscernible over c , then also $\models \theta(a_i, c)$ for all $i < n$, hence $\varphi(x, a_i) \in p$.

We show that strong ULCFS follows from the more explicit form of UDTFS established for NIP formulas in [EK21, BKS24]:

Proposition 9.7. *If a partitioned formula $\varphi(x, y)$ is NIP then it satisfies strong ULCFS.*

Proof. Assume that $\varphi(x, y)$ is NIP. By [EK21, BKS24] there exists $k, m \in \mathbb{N}$ depending only on φ (more precisely, only on the VC-dimension $\text{VC}(\varphi)$) satisfying the following: for every finite set $A \subseteq \mathbb{M}^y$ and every $p \in S_\varphi(A)$, there exist some types $p_1, \dots, p_m \in S_\varphi(A)$ so that:

- (1) each p_i is k -compressible, i.e. for some $A_i \subseteq A$ with $|A_i| \leq k$ we have $p_i \upharpoonright_{A_i} \vdash p_i$,
- (2) and p is the rounded average of the p_i 's, i.e. for any $a \in A$ and $t \in \{0, 1\}$, $\varphi^t(x, a) \in p \iff |\{i \in [m] : \varphi^t(x, a) \in p_i\}| > \frac{m}{2}$.

Let $\Delta(y, (y_j : j \in [k]))$ be the finite set of formulas

$$\left\{ \forall x \left(\left(\bigwedge_{j \in [k]} \varphi^{t_j}(x, y_j) \right) \rightarrow \varphi^t(x, y) \right) : (t_j : j \in [k]) \in \{0, 1\}^{[k]}, t \in \{0, 1\} \right\}.$$

Now let $A \subseteq \mathbb{M}^y$ be an arbitrary finite set and $b \in \mathbb{M}^x$, and let $p := \text{tp}_\varphi(b/A)$. Let $A_i \subseteq A$ and $p_i \in S_\varphi(A)$ for $i \in [m]$ satisfy (1) and (2) above with respect to p , and let $A' := \bigcup_{i \in [m]} A_i$. Then $|A'| \leq km$. We claim that p does not (φ, Δ, n, k) -divide over A' for any $n, k \geq 2$.

For each $i \in [m]$, fix an arbitrary enumeration $A_i = (a_{i,j} : j \in [k])$. Let $a \in A$ be arbitrary, and let $t \in \{0, 1\}$ be such that $\varphi^t(x, a) \in p$. Then by (2) there exists some $s \subseteq [m]$ with $|s| > \frac{m}{2}$ so that $\varphi^t(x, a) \in p_i$ for each $i \in s$. Fix an arbitrary $i \in s$, and for $j \in [k]$ let t_j be such that $\varphi^{t_j}(x, a_{i,j}) \in p_i$. Then by (1) we have

$$\models \forall x \left(\left(\bigwedge_{j \in [k]} \varphi^{t_j}(x, a_{i,j}) \right) \rightarrow \varphi^t(x, a) \right).$$

Let $(a^\ell : \ell \in [n])$ be an arbitrary $\Delta(A')$ -indiscernible sequence in \mathbb{M}^y with $a^1 = a$. In particular $(a^\ell : \ell \in [n])$ is $\Delta(a_{i,j} : j \in [k])$ -indiscernible, hence by the choice of Δ for any $\ell \in [n]$ we also have

$$\models \forall x \left(\left(\bigwedge_{j \in [k]} \varphi^{t_j}(x, a_{i,j}) \right) \rightarrow \varphi^t(x, a^\ell) \right).$$

Let $b_i \in \mathbb{M}^x$ be an arbitrary realization of p_i , in particular $b_i \models \bigwedge_{j \in [k]} \varphi^{t_j}(x, a_{i,j})$, but then also $b_i \models \bigwedge_{\ell \in [n]} \varphi^t(x, a^\ell)$. \square

9.3. Simple theories satisfy strong ULCFS. From the definition of NTP (Definition 2.17) and compactness we have:

Lemma 9.8. *Assume that $\varphi(x, y)$ is NTP and $k \in \mathbb{N}$. Then there is some $K \in \omega$ such that there do not exist $(a_\eta : \eta \in K^K)$ in \mathbb{M}^y such that $\{\varphi(x, a_{\eta \upharpoonright i}) : i < K\}$ is consistent for every $\eta \in K^K$ and $\{\varphi(x, a_{\eta i}) : i < K\}$ is k -inconsistent for every $\eta \in K^{<K}$.*

Proposition 9.9. *If a partitioned formula $\varphi(x, y)$ is NTP then it satisfies strong ULCFS.*

Proof. This follows by compactness from the local character of dividing in simple theories, but we give an explicit finitization of the argument.

Let $\varphi(x, y)$ and $k \in \omega$ be given.

For any $\psi(x, y) \in L$, finite $\Delta \subseteq L$, $n \in \omega$ and tuple of variables z , the set

$$\Xi_{\psi, \Delta, n}(y; z) := \{(y, z) : \psi(x, y) \text{ is } (\Delta, n, k)\text{-dividing over } z\}$$

is \emptyset -definable. Now let K be as given by Lemma 9.8 for $\varphi(x, y)$.

Let $\Delta_K := \emptyset$, and by reverse induction on $i = K - 1, \dots, 0$ we define

$$\begin{aligned} \Delta_i(y_0, \dots, y_{i-1}; y_i) := \\ \exists y_{i+1} \dots y_{K-1} \left(\exists x \bigwedge_{j=0}^{K-1} \varphi(x, y_j) \wedge \bigwedge_{j=i+1}^{K-1} \Xi_{\varphi, \Delta_j, K}(y_j; y_0, \dots, y_{j-1}) \right). \end{aligned}$$

and let $\Delta := \bigcup_{i < K} \Delta_i$.

Let A be a set and $p(x) \in S_\varphi(A)$. We try to choose by induction on $i < K$ elements a_i from A such that $\varphi(x, a_{i+1}) \in p$ and $\varphi(x, a_{i+1})$ is (Δ, K, k) -dividing over $A_i := (a_j : j < i)$.

Suppose that we succeeded. Now we build a $K^{<K}$ -tree for $\varphi(x, y)$ as follows. As $\Delta_i \subseteq \Delta$, in particular $\varphi(x, a_{i+1})$ is (Δ_i, K, k) -dividing over $(a_j : j < i)$ for all $i < K$. Let $b_\emptyset := a_0$. Then let $b_0 := a_1$. By assumption, $\varphi(x, a_1)$ is (Δ, K, k) -dividing over a_0 , let $(b_j : j < K)$ be a $\Delta_1(a_0)$ -indiscernible sequence witnessing this. By the choice of Δ_1 it follows that for each $j < K$ there are some $(b_{j0^i})_{i < (K-1)}$ such that each $\varphi(x, b_{j0^i})$ is (Δ_i, K, k) -dividing over $(b_{j0^h} : h < i)$ and such that $\{\varphi(x, b_{j0^i}) : i < K - 1\}$ is consistent. This gives the first level of the tree. Continuing in the same way we end up with a tree $(b_\eta : \eta \in K^{<K})$ contradicting the choice of K .

Thus we had to get stuck at some $i < K$, which means that p does not (φ, Δ, K, k) -divide over $(a_j : j < i)$, as required. \square

9.4. ULCFS, resilience, NTP_2 .

Definition 9.10. [BYC14, Definition 4.8] A theory T is *resilient* if every formula $\varphi(x, y) \in L$ is *resilient*, i.e. there do not exist an indiscernible sequence $(a_i : i \in \mathbb{Z})$ and an $\{a_i : i \in \mathbb{Z} \setminus \{0\}\}$ -indiscernible sequence $(b_i : i \in \omega)$ so that $a_0 = b_0$, $\{\varphi(x, a_i) : i \in \mathbb{Z}\}$ is consistent and $\{\varphi(x, b_i) : i \in \omega\}$ is inconsistent (hence k -inconsistent for some $k \in \omega$ by compactness and indiscernibility).

Fact 9.11. *If T is either simple or NIP, then it is resilient. And if T is resilient, then it is NTP_2 [BYC14, Proposition 4.11]. It is open if NTP_2 is equivalent to resilience [BYC14, Question 4.14].*

Problem 9.12. Assume that T is not resilient. Is it possible to find a witness to it as in Definition 9.10, such that moreover $\{\varphi(x, b_i) : i \in \omega\}$ is 2-inconsistent?

This following lemma and corollary appeared first in a preliminary version of [CH14], but were removed in the final version due to an error in its intended application.

Lemma 9.13. *The following are equivalent for a theory T .*

- (1) Every formula $\varphi(x, y)$ with $|x| \leq n$ is resilient.
- (2) For every small set D , every c with $|c| \leq n$ and every sequence $\bar{a} = (a_i)_{i \in \mathbb{Q}}$ indiscernible over cD , if $\bar{b} = (b_i)_{i \in \omega}$ is indiscernible over $a_{\neq 0}D$ and $b_0 = a_0$, then there exists \bar{b}' indiscernible over $ca_{\neq 0}D$ such that $\bar{b}' \equiv_{\bar{a}D} \bar{b}$.
- (3) For every small set D , every c with $|c| \leq n$ and every sequence $\bar{a} = (a_i)_{i \in \mathbb{Q}}$ indiscernible over cD , if $\bar{b} = (b_i)_{i \in \omega}$ is indiscernible over $a_{\neq 0}D$ and $b_0 = a_0$, then there exists \bar{b}' indiscernible over $ca_{\neq 0}D$ such that $\bar{b}' \equiv_{a_0D} \bar{b}$.

Proof. (1) implies (2): Let $\varepsilon > 0$ from \mathbb{Q} be arbitrary. Consider the type $p_\varepsilon(x, a_0) = \text{tp}(c/a_0 a_{\notin(-\varepsilon, \varepsilon)}D)$. By indiscernibility of \bar{a} over cD it follows that $\bigcup_{i \in (-\varepsilon, \varepsilon)} p_\varepsilon(x, a_i)$ is consistent (realised by c). We claim that $\bigcup_{i \in \omega} p_\varepsilon(x, b_i)$ is consistent. Otherwise there is some $\varphi(x, a_0 a' d) \in p_\varepsilon(x, a_0)$ with $a' \subseteq a_{\notin(-\varepsilon, \varepsilon)}$ and $d \subseteq D$ such that $\{\varphi(x, b_i a' d)\}_{i \in \omega}$ is inconsistent. Let $a'_i = a_i a' d$ and $b'_i = b_i a' d$. Then $\bar{a}' = (a'_i)_{i \in (-\varepsilon, \varepsilon)}$ is indiscernible, $\bar{b}' = (b'_i)_{i \in \omega}$ is indiscernible over $a'_{\neq 0}$, $a'_0 = b'_0$ and they witness that $\varphi(x, y)$ is not resilient — a contradiction.

As $\varepsilon > 0$ was arbitrary, it follows by compactness that, letting $p(x, a_0) = \text{tp}(c/a_0 a_{\neq 0}D) = \bigcup_{\varepsilon > 0} p_\varepsilon(x, a_0)$, we have that $\bigcup_{i \in \omega} p(x, b_i)$ is consistent. But this means that there is $\bar{b}' \equiv_{\bar{a}D} \bar{b}$ which is indiscernible over $ca_{\neq 0}D$ (see [CH14, Lemma 3.5(1)]).

(2) implies (3) is obvious.

(3) implies (1): Assume that $\varphi(x, y)$ is not resilient, where $|x| \leq n$, and let $\bar{a} = (a_i)_{i \in \mathbb{Q}}, \bar{b} = (b_i)_{i \in \omega}$ witness this. Then by Ramsey, indiscernibility and compactness we may find $c \models \{\varphi(x, a_i)\}_{i \in \mathbb{Q}}$ such that in addition \bar{a} is indiscernible over c . But as $c \models \varphi(x, b_0)$ and $\{\varphi(x, b_i)\}_{i \in \omega}$ is inconsistent, there is no way to make it c -indiscernible keeping the type of \bar{b} even just over the first element. \square

Corollary 9.14. *If T is not resilient, then there is some $\varphi(x, y)$ with $|x| = 1$ which is not resilient.*

Proof. We show by induction that if T satisfies 9.13(2) for $n - 1 \geq 1$, then it satisfies it for n . So let $c = (c_i)_{i < n}$, \bar{a} and \bar{b} be given, with \bar{a} indiscernible over c and \bar{b} indiscernible over $a_{\neq 0}$. Then by the induction hypothesis and Lemma 9.13 with $D = \emptyset$ we find $\bar{b}' \equiv_{\bar{a}} \bar{b}$ and indiscernible over $c_{< n-1} a_{\neq 0}$. Again by the inductive assumption and Lemma 9.13 with $D = c_{< n-1}$ we find a $c_{n-1} c_{< n-1} a_{\neq 0}$ -indiscernible $\bar{b}'' \equiv_{\bar{a}} \bar{b}' \equiv_{\bar{a}} \bar{b}$ — as wanted. \square

Proposition 9.15. (1) *If $\varphi(x, y)$ satisfies ULCFS, then $\varphi(x, y)$ is NTP_2 . In particular, if T satisfies ULCFS for 1-types then T is NTP_2 .*

- (2) If $\varphi(x, y)$ satisfies strong ULCFS then $\varphi(x, y)$ is resilient. In particular, if T satisfies strong UCLFS for 1-types then T is resilient.

Proof. (1) Assume $\varphi(x, y)$ has $k\text{-TP}_2$ for some $k \in \omega$, witnessed by a strongly indiscernible array $(a_{i,j})_{i,j \in \omega}$ (i.e. the rows $\bar{a}_i := (a_{i,j} : j \in \omega), i \in \omega$ are mutually indiscernible, and the sequence of rows $(\bar{a}_i : i \in \omega)$ is indiscernible, see [Che14, Definition 1.1]). Let Δ, n be arbitrary. Let $A := \{a_{i,j} : i, j < n+1\}$, $b \models \{\varphi(x, a_{0,j}) : j < n+1\}$ and $p := \text{tp}_\varphi(b/A)$. Let $C \subseteq A$ be arbitrary with $|C| \leq n$. Then there is some $i < n+1$ so that $\{a_{i,j} : j < n+1\} \cap C = \emptyset$, $(a_{i,j} : j < n+1)$ is indiscernible over C (by strong indiscernibility of the array) and $\{\varphi(x, a_{i,j}) : j < n+1\}$ is k -inconsistent. This shows that p is internally (φ, Δ, n, k) -dividing over C . For the ‘‘in particular’’ part, if T is not NTP_2 , there is a formula $\varphi(x, y)$ with $|x| = 1$ with TP_2 (by Fact 2.19(2)).

(2) Assume $\varphi(x, y)$ is not resilient (if T is not resilient, by Corollary 9.14 there is some non-resilient $\varphi(x, y) \in L$ with $|x| = 1$), then there exist some indiscernible sequence $\bar{b} = (b_i : i \in \mathbb{Z})$ and some $(b_i : i \in \mathbb{Z} \setminus \{0\})$ -indiscernible sequence $\bar{b}' = (b'_i : i \in \omega)$ with $b_0 = b'_0$ so that $\{\varphi(x, b_i) : i \in \mathbb{Z}\}$ is consistent and $\{\varphi(x, b'_i) : i \in \omega\}$ is k -inconsistent for some $k \in \omega$. Let now Δ and $n \in \omega$ be arbitrary. Let $B_n := (b_i : 0 \leq i < n+1)$ and let $p_n \in S_\varphi(B_n)$ be arbitrary so that we have $\{\varphi(x, b_i) : 0 \leq i < n+1\} \subseteq p_n$. We claim that p_n is (φ, Δ, n, k) -dividing over every subset C of B_n with $|C| \leq n$. Indeed, given such C , there is some $0 \leq j < n+1$ with $b_j \in B_n \setminus C$. By indiscernibility of \bar{b} , taking an automorphism σ of \mathbb{M} sending $(\dots, b_{-1}, b_0, b_1, \dots)$ to $(\dots, b_{j-1}, b_j, b_{j+1}, \dots)$ we consider the sequence $\bar{b}'' := \sigma(\bar{b}')$. In particular it is indiscernible over $\{b_i : i \in \mathbb{Z} \setminus \{j\}\}$ and witnesses that $\varphi(x, b_j)$ is (Δ, n, k) -dividing over C . \square

Remark 9.16. We note a weak converse to Proposition 9.15(2): given $\varphi(x, y)$, if there is $k \in \omega$ so that for all Δ, n there is a finite set A with $|A| = n+1$ and $p \in S_\varphi(A)$ so that p is (φ, Δ, n, k) -dividing over every $C \subseteq A$ with $|C| \geq n$, then $\varphi(x, y)$ is not resilient. Indeed, in this case we can find a witness to φ being not resilient by Ramsey and compactness.

The above observations motivate the following conjecture:

Conjecture 9.17. *If $\varphi(x, y)$ is NTP_2 (or even T is NTP_2), then $\varphi(x, y)$ satisfies ULCFS. If $\varphi(x, y)$ is resilient (or even T is resilient), then $\varphi(x, y)$ satisfies strong ULCFS.*

9.5. Counting partial types over finite sets. In this section we consider some finitary versions of the two-cardinal function $f_T(\kappa, \lambda)$ counting partial types from Section 10, which also generalizes the shatter function from VC-theory. In the same way as UDTFS is a strengthening of the Sauer-Shelah lemma, we view ULCFS as a strengthening of this type-counting function being polynomially bounded.

Definition 9.18. Let $\varphi(x, y)$ be a partitioned formula.

- (1) For a set $A \subseteq \mathbb{M}^y$ and $k \in \omega$, we let $S_{\varphi,k}(A)$ be the set of *positive* partial φ -types of size $\leq k$ over A , i.e. consistent sets of formulas of the form $\{\varphi(x, a_i) : i < k\}$ with $a_0, \dots, a_{k-1} \in A$.
- (2) For $p, q \in S_{\varphi,k}(A)$ and $m \in \omega$, we say that p and q are *m-inconsistent* if there are some $p_0 \subseteq p, q_0 \subseteq q$ with $|p_0|, |q_0| \leq m$ so that $p_0 \cup q_0$ is inconsistent.

(3) For $m \leq k \leq l < \omega$, we define

$$f_\varphi(m, k, l) := \max \left\{ |S| : A \subseteq \mathbb{M}^y, S \subseteq S_{\varphi, k}(A), |A| = l, \right. \\ \left. p \neq q \in S \Rightarrow p(x), q(x) \text{ are } m\text{-inconsistent} \right\}.$$

(4) We let $f_\varphi(k, l) := f_\varphi(1, k, l)$ and $f'_\varphi(k, l) := f_\varphi(k, k, l)$.

(5) We will say that $f_\varphi(k, l)$ (or $f'_\varphi(k, l)$) is *polynomially bounded* if there is d such that: for every fixed k there is some $C_k \in \omega$ with $f_\varphi(k, l) \leq C_k l^d$ for all l .

(6) For a formula $\varphi(x, y)$ and $t \in \omega$, we let

$$\left(\bigwedge_t \varphi \right) (x; y_0, \dots, y_{t-1}) := \bigwedge_{i < t} \varphi(x, y_i).$$

Remark 9.19. While the restriction to positive types is natural for our considerations here, we note that for any $\varphi(x, y) \in L$, taking $\psi(x; y_1, y_2) := \varphi(x, y_1) \wedge \neg \varphi(x, y_2) \in L$, each φ -type of size $\leq k$ over a set $A \subseteq \mathbb{M}$ is equivalent to a positive ψ -type of size $\leq k$ over $A \times A$ (more generally, see e.g. [She90, Lemma II.2.1]) — hence many type counting questions transfer to general types.

Remark 9.20. (1) $f_\varphi(m, k, l) \leq f_\varphi(m', k', l')$ and $f_\varphi(m, k, l) \leq f_{\bigwedge_2 \varphi}(m, k, l)$ for all $m' \geq m, k' \geq k, l' \geq l$; in particular $f_\varphi(k, l) \leq f'_\varphi(k, l) \leq \sum_{i \leq k} \binom{l}{i} \leq \left(\frac{e}{k}\right)^k l^k$ for all k, l .
 (2) $f_{\bigwedge_2 \varphi}(m, k, l) \leq f_\varphi(2m, 2k, 2l)$ and $f_\varphi(m, k, l) \leq f_{\bigwedge_m \varphi}(1, k^m, l^m)$ for all m, k, l .

Proof. (1) is immediate from the definition. For the first inequality in (2), let $A \subseteq \mathbb{M}^{y_1, y_2}$ with $|A| = l$ and $S \subseteq S_{\bigwedge_2 \varphi, k}(A)$ a set of pairwise m -inconsistent positive $\bigwedge_2 \varphi$ -types. Let $A' \subseteq \mathbb{M}^y$ be the set $\pi_1(A) \cup \pi_2(A)$, where π_i is the projection onto i th coordinate, then $|A'| \leq 2l$. And for $p \in S$, we let $p' \in S_{\varphi, 2k}(A')$ be defined via: $\varphi(x, a) \in p'$ if and only if there is some a' so that either $\varphi(x, a) \wedge \varphi(x, a') \in p$ or $\varphi(x, a') \wedge \varphi(x, a) \in p$. It follows that for any $p_1 \neq p_2 \in S$, p'_1 and p'_2 are $2m$ -inconsistent. For the second inequality, given a φ -type p of size k over $A \subseteq \mathbb{M}^y$, consider the $\bigwedge_m \varphi$ -type $(\bigwedge_m p)(x)$ of size k^m over A^m given by $\{\bigwedge_{i < m} \varphi(x, a_i) : (a_1, \dots, a_m) \in A^m, \varphi(x, a_i) \in p \text{ for all } i < m\}$ — if p_1, p_2 are m -inconsistent, then p'_1, p'_2 are 1-inconsistent. \square

Proposition 9.21. (1) Assume that $f_{\bigwedge_t \varphi}(k, l)$ is polynomially bounded for all $t \in \omega$, then $\varphi(x, y)$ is NTP₂. In particular, if f_φ is polynomially bounded for all partitioned formulas $\varphi(x, y) \in L$ with $|x| = 1$ in T , then T is NTP₂.
 (2) Assume that $f'_\varphi(k, l)$ is polynomially bounded. Then $f'_{\bigwedge_t \varphi}(k, l)$ is also polynomially bounded for all $t \in \omega$, and $\varphi(x, y)$ is low (see Definition 2.24).

Proof. (1) Assume that $\varphi(x, y)$ has TP₂. Then, by Fact 2.19(2), for some $t = t(\varphi)$ the formula $\psi(x; y_0, \dots, y_{t-1}) := \bigwedge_{i < t} \varphi(x, y_i)$ has 2-TP₂, i.e. there is an array $(a_{i,j})_{i,j < \omega}$ such that $\{\psi(x, a_{i,j}) : j < \omega\}$ is 2-inconsistent for every $i < \omega$ and $\{\psi(x, a_{i,f(i)}) : i < \omega\}$ is consistent for every $f : \omega \rightarrow \omega$.

Let k and l be arbitrary, and let $A := \{a_{i,j}\}_{i < k, j < l}$. For $f : k \rightarrow l$, let $p_f(x) := \{\psi(x, a_{i,f(i)}) : i < k\}$. Then any two types in the set $S := \{p_f : f \in l^k\} \subseteq S_{\psi, k}(A)$ are 1-inconsistent. It follows that $|S| \geq l^k$ while $|A| = k \times l$. That is, $|S| \geq c |A|^k$

with $c = \frac{1}{k^k}$. It follows that for k fixed and $l \gg k$, $f_\psi(k, l)$ grows faster than any polynomial of degree $\leq k - 1$. Thus $f_\psi = f_{\bigwedge_t \varphi}$ is not polynomially bounded.

(2) By Remark 9.20, $f'_{\bigwedge_2 \varphi}(k, l) \leq f'_\varphi(2k, 2l)$, so if the latter is polynomially bounded by $C_k l^d$ then the former is also polynomially bounded by $2^d C_{2k} l^d$.

Assume $\varphi(x, y)$ is not low. Then for an arbitrary large $k \in \omega$ we can find an indiscernible sequence $(a_i : i \in \omega)$ in \mathbb{M}^y so that $\{\varphi(x, a_i) : i \in \omega\}$ is k -consistent, but $(k + 1)$ -inconsistent. But then for every $k \leq l \in \omega$,

$$\left\{ \{\varphi(x, a_{i_{t_j}}) : j < k\} : t_0 < \dots < t_{k-1} < l \right\}$$

is a family of $\geq \binom{l}{k}$ pairwise-inconsistent positive φ -types of size k over $(a_i : i < l)$. Hence $f'_\varphi(k, l) \geq \frac{1}{k^k} l^k$, so f'_φ is not polynomially bounded. \square

Remark 9.22. Since there are non-low (super-)simple theories (see [CK98]), and simplicity of T implies f_φ is polynomially bounded for all formulas (by Propositions 9.9 and 9.23), we see that $f'_\varphi(k, l)$ is polynomially bounded is a strictly stronger condition than $f_\varphi(k, l)$ is polynomially bounded.

Next we observe that, closing under conjunctions, ULCFS implies polynomial boundedness of f_φ .

Proposition 9.23. *For a partitioned formula $\varphi(x, y)$, if $\bigwedge_2 \varphi$ satisfies ULCFS, then $f_\varphi(k, l)$ is polynomially bounded.*

Proof. As $\bigwedge_2 \varphi$ satisfies ULCFS, there is some finite set of formulas $\Delta \subseteq L$ and some $d \in \omega$ so that: for every finite set $B \subseteq \mathbb{M}^{y_1, y_2}$ and every $p'(x) \in S_{\bigwedge_2 \varphi}(B)$ there is some $C \subseteq B$ with $|C| \leq d$ such that p' does not internally $(\bigwedge_2 \varphi, \Delta, d, 2)$ -divide over C .

Let now $A \subseteq \mathbb{M}^y$ be a finite set of size l , and $S \subseteq S_{\varphi, k}(A)$. Given $p \in S$, we consider $\bigwedge_2 p \in S_{\bigwedge_2 \varphi, k^2}(A^2)$ as in Remark 9.20. By the previous paragraph, there is some $A_p \subseteq A^2$ with $|A_p| \leq d$ so that $\bigwedge_2 p$ does not internally $(\bigwedge_2 \varphi, \Delta, d, 2)$ -divide over A_p .

Claim 9.24. There is some $K = K(d, k, \Delta) \in \omega$ satisfying the following. For any $C \subseteq A^2$ with $|C| = d$, if $(p_i(x) : i < n)$ with $p_i \in S_{\varphi, k}(A)$ are pairwise 1-inconsistent and $\bigwedge_2 p_i$ does not internally $(\bigwedge_2 \varphi, \Delta, d, 2)$ -divide over C for all $i < n$, then $n \leq K$.

Proof. For $i < n$, let $p_i = \{\varphi(x, a_{i,t}) : t < k\}$ for some $a_{i,t} \in A$, and let $\bar{a}_i := (a_{i,t} : t < k)$. By Ramsey's theorem, using pairwise 1-inconsistency of the p_i 's, there is some $K = K(d, k, \Delta)$ satisfying the following. If $n > K$ then there is a subsequence $(\bar{a}_{i_j} : j < d)$ of $(\bar{a}_i : i < n)$ which is Δ -indiscernible over C and some $\alpha, \beta < k$ so that for all $j < j' < d$ we have: $\varphi(x, a_{i_j, \alpha}) \in p_{i_j}, \varphi(x, a_{i_{j'}, \beta}) \in p_{i_{j'}}$ and $\{\varphi(x, a_{i_j, \alpha}), \varphi(x, a_{i_{j'}, \beta})\}$ is inconsistent. This shows that $\varphi(x, a_{i_j, \alpha}) \wedge \varphi(x, a_{i_{j'}, \beta}) \in \bigwedge_2(p_{i_0})$ internally $(\Delta, d, 2)$ -divides over C — a contradiction. \square

As there are at most $(|A^2|)^d \leq l^{2d}$ choices for A_p , it follows from the claim that for any k , any family $S \subseteq S_{\varphi, k}(A)$ of pairwise 1-inconsistent types has size at most Kl^{2d} , with K depending only on φ and k — hence $f_\varphi(k, l)$ is polynomially bounded. \square

Conjecture 9.25. *The following are equivalent for a partitioned formula $\varphi(x, y)$:*

- (1) $\bigwedge_t \varphi$ is NTP_2 for all $t \in \omega$;
- (2) $f_{\bigwedge_t \varphi}(k, l)$ is polynomially bounded for all $t \in \omega$.

Or at least: $f_\varphi(k, l)$ is polynomially bounded for every formula $\varphi(x, y)$ in an NTP_2 theory T .

We note that it is necessary to consider conjunctions in Conjecture 9.25:

Example 9.26. Let T the theory of an infinite triangle-free random graph. Then the edge relation $R(x, y)$ is NTP (hence satisfies strong ULCFS by Proposition 9.9), while the formula $R(x, y_1) \wedge R(x, y_2)$ has 2-TP_2 (see [Che14, Example 3.13]) — hence does not satisfy ULCFS by Proposition 9.15, and $f_{\bigwedge_2 R}(k, l)$ is not polynomially bounded by (the proof of) Proposition 9.21(1). Note that $f_R(k, l) = f'_R(k, l)$ (since in a triangle-free random graph, two partial types $\{R(x, a_i) : i < m\}$ and $\{R(x, b_j) : j < n\}$ are inconsistent if and only if $\models R(a_i, b_j)$ for some i, j , in which case $\{R(x, a_i), R(x, b_j)\}$ is inconsistent), hence $f_R(k, l)$ is also not polynomially bounded by Proposition 9.21(2).

Next we observe that at least TP_2 is characterized by the maximality of the function $f_\varphi(k, l)$ (unlike in the infinitary version of the problem, see Corollary 10.18). We use a classical result of Erdős:

Fact 9.27. [Erd64] *Let $G = (V, E)$ be a uniform k -hypergraph (i.e. $E \subseteq \binom{V}{k}$) with $|V| = l$ such that it does not contain a complete k -partite uniform k -hypergraph $K_{d, \dots, d}$ with each part of size d . Then the number of hyper-edges for all $l \gg k, d$ satisfies $|E| \leq l^{k - \frac{1}{d^{k-1}}}$.*

Proposition 9.28. *The following are equivalent.*

- (1) The partitioned formula $\varphi(x, y)$ does not have 2-TP_2 .
- (2) The function $f_\varphi(k, l)$ has power saving, i.e. there exists some $\varepsilon = \varepsilon(\varphi) > 0$ and $k_0 \in \mathbb{N}$ so that: for each $k \geq k_0$ there is some $C = C(\varphi, k) \in \mathbb{N}$ so that $f_\varphi(k, l) \leq Cl^{k-\varepsilon}$ for all l .

Proof. (2) \Rightarrow (1). The proof of Proposition 9.21(1) shows that if $\varphi(x, y)$ has 2-TP_2 then for any $k \in \omega$ and $l \geq k$, $f_\varphi(k, l) \geq \left(\frac{1}{k}\right)^k l^k$, which grows with l faster than $Cl^{k-\varepsilon}$ for any C and $\varepsilon > 0$.

(1) \Rightarrow (2): If $\varphi(x, y)$ does not have 2-TP_2 , then by compactness there exist some $1 < d \in \omega$ such that there are no $(a_{i,j} : 1 \leq i, j \leq d)$ so that $\{\varphi(x, a_{i,j}) : 1 \leq j \leq d\}$ is 2-inconsistent for all $1 \leq i \leq d$ and $\{\varphi(x, a_{i,f(i)}) : 1 \leq i \leq d\}$ is consistent for every $f : [d] \rightarrow [d]$.

Suppose that no $\varepsilon > 0$ and k_0 satisfy (2). In particular, taking $\varepsilon := \frac{1}{d^{d-1}}$, there exists some $k \geq d$ so that: for any $C \in \mathbb{N}$ there is an arbitrarily large $l \in \mathbb{N}$ with $f_\varphi(k, l) > Cl^{k-\varepsilon}$. Let $C := 2$, and, as $\varepsilon < 1$, choose l sufficiently large so that $\left(\frac{e}{k-1}\right)^{k-1} l^{k-1} \leq l^{k-\varepsilon}$.

Let $S \subseteq S_{\varphi, k}(A)$ with $|S| > Cl^{k-\varepsilon}$ and $A \subseteq \mathbb{M}^y, |A| = l$ be a family of pairwise 1-inconsistent types witnessing this. Note that the number of partial types $p \in S$ of size $< k$ is at most $\sum_{i \leq k-1} \binom{l}{i} \leq \left(\frac{e}{k-1}\right)^{k-1} l^{k-1} \leq l^{k-\varepsilon}$ by assumption on l .

Hence there is $S' \subseteq S$ with $|S'| > l^{k-\varepsilon}$ so that all $p \in S'$ are of size *exactly* k .

Consider the uniform k -hypergraph G with

$$V(G) := A, E(G) := \{\text{dom}(p) : p \in S'\} \quad (\text{so } |E| \geq l^{k-\varepsilon}).$$

Assume that for every $\bar{a} = \{a_{d+1}, \dots, a_k\} \in \binom{V}{k-d}$ we have $|E_{\bar{a}}| \leq l^{d-\varepsilon}$, where $E_{\bar{a}} = \{\{a_1, \dots, a_k\} \in \binom{V}{k} : \{a_1, \dots, a_k\} \cup \bar{a} \in E\}$. As there are $\binom{l}{k-d} \leq l^{k-d}$ choices for \bar{a} , this would imply that $|E| \leq l^{k-\varepsilon}$ — contradicting the assumption. Hence there is some $\bar{a} \in \binom{V}{k-d}$ so that the d -hypergraph $G_{\bar{a}} := (V, E_{\bar{a}})$ has $> l^{d-\varepsilon}$ hyperedges. By Fact 9.27, $G_{\bar{a}}$ contains a complete d -partite d -hypergraph H with each part of size d . For $1 \leq i \leq d$, let $\bar{a}_i := (a_{i,j} : 1 \leq j \leq d)$ with $a_{i,j} \in A$ enumerate the i 's part of the partition of H .

As each hyperedge of E corresponds to a domain of some $p \in S'$, it follows that

$$\{\varphi(x, a_{i,f(i)}) : 1 \leq i \leq d\} \cup \{\varphi(x, a_{d+1}), \dots, \varphi(x, a_k)\}$$

is consistent for each $f : [d] \rightarrow [d]$. On the other hand, let $1 \leq i' \leq d$ and $1 \leq j \neq j' \leq d$ be arbitrary. As H is complete, it follows from its definition that

$$\begin{aligned} \{\varphi(x, a_{i,1}) : 1 \leq i \leq d, i \neq i'\} \cup \{\varphi(x, a_{i',j})\} \cup \{\varphi(x, a_{d+1}), \dots, \varphi(x, a_k)\} &\in S', \\ \{\varphi(x, a_{i,1}) : 1 \leq i \leq d, i \neq i'\} \cup \{\varphi(x, a_{i',j'})\} \cup \{\varphi(x, a_{d+1}), \dots, \varphi(x, a_k)\} &\in S'. \end{aligned}$$

As any two types in S' are 1-inconsistent, the only possibility is that

$$\{\varphi(x, a_{i',j}), \varphi(x, a_{i',j'})\}$$

is inconsistent. It follows that $\{\varphi(x, a_{i,j}) : 1 \leq j \leq d\}$ is 2-inconsistent for every $1 \leq i \leq d$. But then $(a_{i,j} : 1 \leq i, j \leq d)$ contradicts the choice of d \square

Hence we have the following weak partial result towards Conjecture 9.25:

Corollary 9.29. *The following are equivalent for a partitioned formula $\varphi(x, y)$:*

- (1) $\bigwedge_t \varphi$ is NTP₂ for all $t \in \mathbb{N}$;
- (2) $f_{\bigwedge_t \varphi}(k, l)$ satisfies power saving for all $t \in \mathbb{N}$.

Proof. By Proposition 9.28 and Fact 2.19(2). \square

Problem 9.30. (1) Is there an implication in either direction between “ $\varphi(x, y)$ is FHP” and “ $f'_\varphi(k, l)$ is polynomially bounded”? At least at the level of the theory T as opposed to individual formula? Matousek’s theorem (Fact 2.22), combined with Remark 9.19, shows that if there is some C, d so that $f'_\varphi(l, l) \leq Cl^d$ for all l then $\varphi(x, y)$ is FHP — so part of the question is if this assumption could be relaxed to polynomial boundedness of $f'_\varphi(k, l)$.
 (2) Does FHP imply that $f_\varphi(k, l)$ is polynomially bounded? By Proposition 2.21 this would follow from Conjecture 9.25.

Problem 9.31. We can also ask a more general version of Conjecture 9.25, corresponding to a finitary version of Conjecture 10.6: what are the possible functions $f_\varphi(k, l)$?

10. COUNTING PARTIAL TYPES IN THE INFINITARY SETTING

Definition 10.1. For a complete first-order theory T , $1 \leq n \in \omega$ and an infinite cardinal κ , we let $f_T^n(\kappa) := \sup\{|S_n(M)| : M \models T, |M| = \kappa\}$ and $f_T(\kappa) := \sup_{n \in \omega} f_T^n(\kappa)$.

A celebrated result of Keisler [Kei74, Kei78] (see also [Kei76]), refining earlier work of [She71] and Morley [Mor65], demonstrates that there are exactly six possibilities for f_T when T is a complete countable theory:

Fact 10.2. *Let T be a complete countable first-order theory with an infinite model (we will assume this throughout the section). Then for all $\kappa \geq \aleph_0$ we have $f_T(\kappa) = f_T^1(\kappa)$ and:*

- (1) T is ω -stable $\Rightarrow f_T(\kappa) = \kappa$;
- (2) T is superstable, not ω -stable $\Rightarrow f_T(\kappa) = \kappa + 2^{\aleph_0}$;
- (3) T is stable, not superstable $\Rightarrow f_T(\kappa) = \kappa^{\aleph_0}$;
- (4) T is not multiorder, unstable $\Rightarrow f_T(\kappa) = \text{ded } \kappa$;
- (5) T is NIP, multiorder $\Rightarrow f_T(\kappa) = (\text{ded } \kappa)^{\aleph_0}$;
- (6) T is not NIP $\Rightarrow f_T(\kappa) = 2^\kappa$.

Here $\text{ded } \kappa$ is the supremum over all cardinals λ so that there exists a linear order of size κ with λ many Dedekind cuts (equivalently, the supremum of the sizes of linear orders with a dense subset of size at most κ), and for the definition of the multiorder property see Definition 10.10. We always have $\kappa < \text{ded } \kappa \leq 2^\kappa$, hence under the generalized continuum hypothesis (GCH), $\text{ded } \kappa = 2^\kappa$ for all infinite κ , in which case the functions in (4)–(6) are equal. It is also consistent with ZFC that the functions in (5) and (6) are distinct [Mit72]; it is also consistent that the functions in (4) and (5) are distinct [CKS16]. But it is not known if it is consistent that all three functions in (4), (5), (6) are pairwise distinct simultaneously. Yet, in ZFC, for every infinite κ , we have $2^\kappa \leq \text{ded } \text{ded } \text{ded } \text{ded } \kappa$ [CS16]. We refer to [CKS16, CS16] for a further discussion of this function.

Here we consider the following two-cardinal refinement of f_T introduced in [She80] in order to characterize simple theories, and further studied in [Cas99, Les00, Cas03] (where it is denoted as $\text{NT}(\kappa, \lambda)$).

Definition 10.3. For a complete first-order theory T and infinite cardinals $\kappa \leq \lambda$, we let $f_T^n(\kappa, \lambda)$ be the supremum of the cardinalities $|P|$, where P is a family of pairwise inconsistent partial n -types each of cardinality $\leq \kappa$, all over the same fixed set of parameters of size λ . We let $f_T(\kappa, \lambda) := \sup_{n \in \omega} f_T^n(\kappa, \lambda)$.

The following is immediate from the definitions:

Remark 10.4. In any theory T we have:

- (1) $f_T^n(\lambda) = f_T^n(\lambda, \lambda)$ and $f_T^n(\kappa, \lambda) \leq f_T^{n+1}(\kappa, \lambda)$ for $\kappa \leq \lambda \geq |T|$ and $n \in \omega$;
- (2) $f_T^1(1, \lambda) \geq \lambda$ for all $\lambda \geq \aleph_0$;
- (3) $f_T^n(\kappa, \lambda) \leq f_T^n(\kappa', \lambda')$ for any $\lambda' \geq \lambda, \kappa' \geq \kappa$;
- (4) $f_T(\kappa, \lambda) \leq \lambda^\kappa$ for all $\lambda \geq \kappa \geq \aleph_0$.

This two-cardinal function is known to characterize simplicity and supersimplicity of T as follows (restricting to countable theories):

Fact 10.5. *Let T be a complete countable theory with infinite models, and $\kappa \leq \lambda$ infinite cardinals.*

- (1) *The following are equivalent:*
 - (a) T is simple;
 - (b) $f_T(\kappa, \lambda) \leq \lambda^{\aleph_0} + 2^\kappa$ for all κ, λ ;
 - (c) $f_T^1(\kappa, \lambda) < \lambda^\kappa$ for some $\lambda = \lambda^{<\kappa}$.
- (2) *The following are equivalent:*
 - (a) T is supersimple;
 - (b) $f_T(\kappa, \lambda) \leq \lambda + 2^\kappa$ for all κ, λ ;
 - (c) $f_T^1(\kappa, \lambda) < \lambda^{\aleph_0}$ for some κ, λ .

Proof. (1) The equivalence of (a), (b) and (c) is from [She80, Theorem 0.2] (see also [She90, Theorem III.7.7] or [Kim96, Proposition 2.20]) and [Cas99, Theorem 2.8]. Item (2) is from [Cas99, Theorem 3.2]. \square

Motivated by Facts 10.2 and 10.5, Adler made the following conjecture:

Conjecture 10.6. [Adl09]

- (1) *There are only finitely many possible functions $f_T(\kappa, \lambda)$ when T is a countable theory T .*
- (2) *The property “ T is NTP_2 ” can be detected from $f_T(\kappa, \lambda)$ (note that if T is TP_2 , then $f_T(\kappa, \lambda)$ is maximal, i.e. equal to λ^κ for all $\aleph_0 \leq \kappa \leq \lambda$).*

In what follows, we will refute Conjecture 10.6(2) and confirm Conjecture 10.6(1) assuming GCH.

Definition 10.7. (1) As usual, by a tree we mean a partial order (\mathcal{T}, \leq) so that for every $t \in \mathcal{T}$, $\{s \in \mathcal{T} : s \leq t\}$ is linearly well-ordered by \leq . By a branch in a tree, we mean a maximal linearly ordered subset of the nodes. The length of a branch is its order type. The level $l(t)$ of $t \in T$ is the order type of $\{s \in \mathcal{T} : s < t\}$. The height of \mathcal{T} is $\sup\{l(t) + 1 : t \in \mathcal{T}\}$.

- (2) For two infinite cardinals κ, λ , we let $\lambda^{\kappa, \text{tr}}$ denote the supremum of cardinals μ so that there exists a tree with $\leq \lambda$ many nodes and $\geq \mu$ many branches of length $\leq \kappa$.

Remark 10.8. (1) Equivalently, $\lambda^{\kappa, \text{tr}}$ can be defined as the supremum of cardinals μ so that there exists a tree of height $\leq \kappa$ and size $\leq \lambda$ with $\geq \mu$ branches (by restricting to the subtree given by the union of all branches of length $\leq \kappa$ with the induced ordering).

- (2) Note that $\lambda \leq \lambda^{\kappa, \text{tr}} \leq \lambda^\kappa$ and if $\lambda = \lambda^{<\kappa}$, then $\lambda^{\kappa, \text{tr}} = \lambda^\kappa$ (witnessed by the tree $\lambda^{<\kappa}$ with the usual tree ordering: for $\eta_i \in \lambda^{<\kappa}$, say $\eta_i \in \lambda^{\alpha_i}$ for ordinals $\alpha_1 \leq \alpha_2 < \kappa$, $\eta_1 \leq \eta_2$ if $\eta_2 \upharpoonright_{\alpha_1} = \eta_1$).
- (3) We have $\kappa^{\kappa, \text{tr}} = \text{ded } \kappa$, by (1) and [Bau76, Theorem 2.1(b)] (see also [CKS16, Section 6.1] for a more detailed discussion).

Proposition 10.9. *Given infinite cardinals $\kappa \leq \lambda$, if $\lambda^\kappa > \lambda + 2^\kappa$ then for some regular $\theta \leq \kappa$ there is a tree with $\leq \lambda$ nodes, θ levels and λ^κ branches.*

In particular, if $\lambda^\kappa > 2^\kappa$ then $\lambda^{\kappa, \text{tr}} = \lambda^\kappa$.

Proof. Let $\mu := \min\{\mu : \mu^\kappa \geq \lambda\} \leq \lambda$. Note that $\mu > 2^\kappa$ (as otherwise $\mu^\kappa \leq 2^\kappa$, hence $\lambda \leq 2^\kappa$, so $\lambda^\kappa \leq 2^\kappa$ — contradicting the assumption). Then $\mu^\kappa = (\mu^\kappa)^\kappa \geq \lambda^\kappa > \lambda + 2^\kappa$. Also for a cardinal ν , $\nu < \mu \Rightarrow \nu^\kappa < \lambda$ by minimality, hence also $\nu < \mu \Rightarrow \nu^\kappa < \mu$ (for finite ν as $\mu > 2^\kappa$, and for infinite ν as otherwise $\nu^\kappa = (\nu^\kappa)^\kappa \geq \mu^\kappa > \lambda$). Let $\theta := \text{cf}(\mu)$ — regular. We know that $\mu^\kappa > \mu$ (as $\mu \leq \lambda$ and $\mu^\kappa > \lambda$). On the other hand, if $\kappa < \theta = \text{cf}(\mu) \leq \mu$ then using the above observations we would get $\mu^\kappa = \mu$ (see e.g. [Kun14, Theorem 5.20(iii)(a)]), a contradiction. So necessarily $\theta \leq \kappa$.

Consider the tree $\mu^{<\theta}$, it has θ levels and μ^θ -many branches of length θ . It also has $\sum_{\beta < \theta \text{ ordinal}} \mu^{|\beta|}$ nodes. Note that for each $\beta < \theta = \text{cf}(\mu)$ we have $\mu^{|\beta|} \leq \mu \cdot \sup\{\nu^{|\beta|} : \nu < \mu \text{ cardinal}\}$ (see e.g. [HSW09, Lemma 1.6.15(c)]), and $\nu^{|\beta|} \leq \nu^\theta \leq \nu^\kappa < \lambda$ by minimality of μ . Hence $\mu^{|\beta|} \leq \mu \cdot \lambda = \lambda$, so the number of nodes in the tree $\mu^{<\theta}$ is at most $\theta \cdot \lambda = \lambda$.

And as $\kappa \geq \theta = \text{cf}(\mu)$ we have (see e.g. [HSW09, Lemma 1.6.15(b)]) $\mu^\kappa = (\sup \{\nu^\kappa : \nu < \mu \text{ cardinal}\})^{\text{cf}(\mu)=\theta} \leq \mu^\theta \leq \lambda^\theta \leq \lambda^\kappa \leq (\mu^\kappa)^\kappa = \mu^\kappa$. So $\mu^\theta = \lambda^\kappa$, and we are done.

For the ‘‘in particular’’ part, note that if $\lambda^\kappa = \lambda$ then clearly $\lambda^\kappa = \lambda^{\kappa, \text{tr}}$; and otherwise $\lambda^\kappa > \lambda + 2^\kappa$ and the result above applies. \square

We recall the multi-order property from [Kei76] (which is equivalent to the theory T not admitting *ird-patterns* of infinite height, or $\kappa_{\text{ird}}(T) > \aleph_0$ in the sense of [She90, Definition III.7.1], see also [Adl07, Section 5]):

Definition 10.10. A theory T has the *multi-order property* if there exist formulas $\varphi_i(x, y_i) \in L$ for $i \in \omega$ with x fixed and y_i arbitrary tuples of variables, and tuples $(a_{i,j} : i, j \in \omega)$ so that: for every $f : \omega \rightarrow \omega$ there exists some b_f with $\models \varphi(b_f, a_{i,j}) \iff j < f(i)$ for all $i, j \in \omega$.

Proposition 10.11. *The multi-order property is witnessed in one variable, i.e. if T is multi-order then we can choose the formulas $\varphi_i(x, y_i)$ in Definition 10.10 with $|x| = 1$.*

Proof. This follows from [Sim22, Proposition 4.4] (see also [GH15] for the analogous sub-additivity statement in the case of finite cardinals). Namely, by Ramsey and compactness it is easy to see that T has the multi-order property witnessed by some formulas $\varphi(x, y_i), i \in \omega$ with $|x| = n$ if and only if it is not the case that $\text{opD}(x = x) < \aleph_0$ in the sense of [Sim22, Definition 4.1], if and only if there is an n -tuple a so that $\text{opD}(a/\emptyset) < \aleph_0$ does not hold. But [Sim22, Proposition 4.4] implies that for any finite tuples a, b , if $\text{opD}(a/\emptyset), \text{opD}(b/\emptyset) < \aleph_0$, then $\text{opD}(ab/\emptyset) < \aleph_0$ — hence the multi-order property reduces to singletons. \square

By analogy with the multi-order property, we define the multi-TP₁ property:

Definition 10.12. We say that T has the *multi-TP₁* property if there exist formulas $\varphi_i(x, y_i) \in L$ with x fixed and y_i arbitrary finite tuples of variables and trees of tuples $(a_\eta^i)_{\eta \in \omega^{<\omega}}$ for $i \in \omega$ in \mathbb{M} such that:

- (1) for any choice of branches $(\eta_i : i \in \omega)$ with $\eta_i \in \omega^\omega$, the set of formulas $\bigcup_{i \in \omega} \{\varphi_i(x; a_{\eta_i|\alpha}^i) : \alpha < \omega\}$ is consistent,
- (2) for all $i \in \omega$ and all $\eta \perp \nu$ in $\omega^{<\omega}$, $\{\varphi_i(x; a_\eta^i), \varphi_i(x; a_\nu^i)\}$ is inconsistent.

Proposition 10.13. *If T is NIP and multi-order then T is multi-TP₁, witnessed by formulas $(\varphi_i(x, y_i))_{i \in \omega}$ with $|x| = 1$.*

Proof. By Proposition 10.11, let $\varphi_i(x, y_i) \in L$ for $i \in \omega$ with $|x| = 1$ witness the multi-order property.

We will use an elaboration of the trick in Shelah’s proof that NIP plus unstable implies the strict order property SOP. By Ramsey and compactness, we can choose tuples $(a_{i,j} : i \in \omega, j \in \mathbb{Q})$ so that for every $f : \omega \rightarrow \mathbb{Q}$ there exists some b_f with $\models \varphi(b_f, a_{i,j}) \iff j < f(i)$ for all $i \in \omega, j \in \mathbb{Q}$; and so that the sequences $\{\bar{a}_i : i \in \omega\}$ with $\bar{a}_i = (a_{i,j} : j \in \mathbb{Q})$ are mutually indiscernible.

By induction on $\alpha < \omega$ we will choose formulas $\varphi'_\alpha(x, y'_\alpha) \in L$ and sequences $\bar{a}'_\alpha = (a'_{\alpha,j} : j \in \mathbb{Q})$ so that:

- (1) $(\bar{a}'_i : i \leq \alpha) \cup (\bar{a}_i : \alpha < i < \omega)$ are mutually indiscernible;

(2) the set of formulas $\pi'_{\leq\alpha}(x) \cup \pi_{>\alpha}(x)$ is consistent, where

$$\begin{aligned}\pi'_{\leq\alpha}(x) &= \bigcup_{i \leq \alpha} (\{\varphi'_i(x, a'_{i,j}) : j \in \mathbb{Q}, j < 0\} \cup \{\neg\varphi'_i(x, a'_{i,j}) : j \in \mathbb{Q}, j \geq 0\}), \\ \pi_{>\alpha}(x) &= \bigcup_{\alpha < i < \omega} (\{\varphi_i(x, a_{i,j}) : j \in \mathbb{Q}, j < 0\} \cup \{\neg\varphi_i(x, a_{i,j}) : j \in \mathbb{Q}, j \geq 0\});\end{aligned}$$

(3) for all $i \leq \alpha$ and $j_2 > j_1 \in \mathbb{Q}$, $\varphi'_i(x, a'_{i,j_2}) \vdash \varphi'_i(x, a'_{i,j_1})$.

Fix $\alpha \in \omega$, and assume we have already chosen $(\varphi'_i(x, y'_\alpha) : i < \alpha)$ and $(\bar{a}'_i : i < \alpha)$ satisfying (1)–(3).

Since the formula $\varphi_\alpha(x, y_\alpha)$ is NIP, there is some $k_\alpha \in \omega$ and $\eta : k_\alpha \rightarrow \{0, 1\}$ so that $\bigwedge_{j \in k_\alpha} \varphi_\alpha(x, a_{\alpha,j})^{\eta(j)}$ is inconsistent (where $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$). Starting with this formula, we change one by one instances of $\neg\varphi_\alpha(x, a_{\alpha,j}) \wedge \varphi_\alpha(x, a_{\alpha,j+1})$ to $\varphi_\alpha(x, a_{\alpha,j}) \wedge \neg\varphi_\alpha(x, a_{\alpha,j+1})$. After finitely many steps, we arrive to a formula of the form $\bigwedge_{j \in \omega, 0 \leq j < \ell} \varphi_\alpha(x, a_{\alpha,j}) \wedge \bigwedge_{j \in \omega, \ell \leq j < k_\alpha} \neg\varphi_\alpha(x, a_{\alpha,j})$ for some $0 \leq \ell < k_\alpha$, and by assumption (2) for $\alpha - 1$ and mutual indiscernibility this latter formula is consistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x)$. Therefore there is a step in the process in which we pass from a formula inconsistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x)$ to a formula consistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x)$. Namely, there is some $j_0 \in k_\alpha$ and $\eta_0 : k_\alpha \rightarrow \{0, 1\}$ so that

$$\bigwedge_{0 \leq j < j_0} \varphi_\alpha(x, a_{\alpha,j})^{\eta_0(j)} \wedge \varphi_\alpha(x, a_{\alpha,j_0}) \wedge \neg\varphi_\alpha(x, a_{\alpha,j_0+1}) \wedge \bigwedge_{j_0+1 < j < k_\alpha} \varphi_\alpha(x, a_{\alpha,j})^{\eta_0(j)}$$

is consistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x)$, but

$$\bigwedge_{0 \leq j < j_0} \varphi_\alpha(x, a_{\alpha,j})^{\eta_0(j)} \wedge \neg\varphi_\alpha(x, a_{\alpha,j_0}) \wedge \varphi_\alpha(x, a_{\alpha,j_0+1}) \wedge \bigwedge_{j_0+1 < j < k_\alpha} \varphi_\alpha(x, a_{\alpha,j})^{\eta_0(j)}$$

is inconsistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x)$, hence already with some finite subtype.

By compactness we can extend (without changing the already given elements) our sequences $\bar{a}'_i = (a'_{i,j} : j \in \mathbb{Q})$ to $(a'_{i,j} : j \in J_- + \mathbb{Q} + J_+)$ for $i < \alpha$ and $\bar{a}_i = (a_{i,j} : j \in \mathbb{Q})$ to $(a_{i,j} : j \in J_- + \mathbb{Q} + J_+)$ for $\alpha \leq i < \omega$, maintaining mutual indiscernibility, where $J_- = \omega$, $J_+ = \omega^*$.

Then, by the above and mutual indiscernibility, there are some finite sets $J_0 \subseteq J_-$, $J_1 \subseteq J_+$ and $I \subseteq \omega$, $\alpha < I$ so that, taking

$$\begin{aligned}\tilde{\pi}'_{\leq\alpha-1}(x; \bar{a}_{\leq\alpha-1, J_0, J_1}) &:= \{\varphi'_i(x, a'_{i,j}) : i \leq \alpha - 1, j \in J_0\} \cup \\ &\quad \{\neg\varphi'_i(x, a'_{i,j}) : i \leq \alpha - 1, j \in J_1\}, \\ \tilde{\pi}_{>\alpha}(x; \bar{a}_{I, J_0, J_1}) &:= \{\varphi_i(x, a_{i,j}) : i \in I, j \in J_0\} \cup \{\neg\varphi_i(x, a_{i,j}) : i \in I, j \in J_1\},\end{aligned}$$

the first formula is consistent with $\pi'_{\leq\alpha-1}(x) \cup \pi_{>\alpha}(x) \cup \tilde{\pi}'_{\leq\alpha-1}(x) \cup \tilde{\pi}_{>\alpha}(x)$, while the second formula is inconsistent with $\tilde{\pi}'_{\leq\alpha-1}(x) \cup \tilde{\pi}_{>\alpha}(x)$. Fix an order preserving bijection $\gamma : \mathbb{Q} \rightarrow (j_0 - 1, j_0 + 2) \cap \mathbb{Q}$ with $\gamma(j_0) = j_0$, $\gamma(j_0 + 1) = j_0 + 1$, and for $j \in \mathbb{Q}$ we consider the finite tuple

$$\begin{aligned}a'_{\alpha,j} &:= a_{\alpha,\gamma(j)} \cup (a_{\alpha,j'} : j' \in \omega, 0 \leq j' < j_0 \vee j_0 + 1 < j' < k_\alpha) \cup \\ &\quad (a'_{i,j'} : i \leq \alpha - 1, j' \in J_0 \cup J_1) \cup (a_{i,j'} : i \in I, j' \in J_0 \cup J_1)\end{aligned}$$

and formula

$$\begin{aligned} \varphi'_\alpha(x, y'_\alpha) &:= \varphi_\alpha(x, y_{\alpha, j_0}) \wedge \bigwedge_{0 \leq j < j_0} \varphi_\alpha(x, y_{\alpha, j})^{\eta_0(j)} \wedge \bigwedge_{j_0+1 < j < k_\alpha} \varphi_\alpha(x, y_{\alpha, j})^{\eta_0(j)} \wedge \\ &\quad \tilde{\pi}'_{\leq \alpha-1}(x; y_{\leq \alpha-1, J_0, J_1}) \wedge \tilde{\pi}'_{> \alpha}(x; y_{I, J_0, J_1}). \end{aligned}$$

By construction, the sequences $\bar{a}'_\alpha, (\bar{a}'_i)_{i < \alpha}, (\bar{a}_i)_{\alpha < i < \omega}$ are still mutually indiscernible (so (1) holds). Using indiscernibility, for all $j_2 > j_1 \in \mathbb{Q}$ we have $\varphi'_\alpha(x, a'_{\alpha, j_2}) \vdash \varphi'_\alpha(x, a'_{\alpha, j_1})$ (so (3) holds), and as $\varphi'_\alpha(x, a'_{\alpha, j_0}) \wedge \neg \varphi'_\alpha(x, a'_{\alpha, j_0+1})$ is consistent with $\pi'_{\leq \alpha-1}(x) \cup \pi_{> \alpha}(x)$ implies that (2) also holds.

We can think of $(\varphi'_i, \bar{a}'_i)_{i \in \omega}$ as witnessing that T satisfies “multi-SOP”. From this we can produce multi-TP₁ similarly to the standard argument that SOP implies TP₁. Namely, by induction on $|\eta|$, we can choose a tree of non-empty closed intervals $[i_\eta, i'_\eta]$ in \mathbb{Q} with $\eta \in \omega^{< \omega}$ so that:

- (a) for all $\nu \leq \eta$ in $\omega^{< \omega}$, $\emptyset \neq [i_\eta, i'_\eta] \subseteq [i_\nu, i'_\nu]$;
- (b) for all $\eta \perp \nu$ in $\omega^{< \omega}$, $[i_\eta, i'_\eta] \cap [i_\nu, i'_\nu] = \emptyset$.

For $\alpha \in \omega$ and $\eta \in \omega^{< \omega}$, let $b_\eta^\alpha := (a'_{\alpha, i_\eta}, a'_{\alpha, i'_\eta})$, and let $\psi_\alpha(x, b_\eta^\alpha) := \varphi'_\alpha(x, a'_{\alpha, i_\eta}) \wedge \neg \varphi'_\alpha(x, a'_{\alpha, i'_\eta})$. It follows that for any $\alpha \in \omega$ and $\eta \perp \nu$ in $\omega^{< \omega}$, $\{\psi_\alpha(x; b_\eta^\alpha), \psi_\alpha(x; b_\nu^\alpha)\}$ is inconsistent by (3) and (b). And for every choice of branches $(\eta_\alpha : \alpha \in \omega)$ with $\eta_\alpha \in \omega^\omega$, $\bigcup_{\alpha \in \omega} \{\psi_\alpha(x; b_{\eta_\alpha|t}^\alpha) : t < \omega\}$ is consistent by (a), (2)+(1) and compactness. \square

The following is a generalization of [Kei78, Lemma 4, (ii) \Rightarrow (iii)].

Proposition 10.14. (1) *If T has TP₁ then $f_T^1(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}}$ for all $\aleph_0 \leq \kappa \leq \lambda$.*
 (2) *If T has multi-TP₁, witnessed by formulas $(\varphi_\alpha(x, y_\alpha))_{\alpha \in \omega}$ with $|x| = 1$, then $f_T^1(\kappa, \lambda) \geq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$ for all $\aleph_0 \leq \kappa \leq \lambda$.*

Proof. (1) By Fact 2.19 some formula $\varphi(x, y)$ with $|x| = 1$ has TP₁, witnessed by $A := (a_\eta : \eta \in \omega^{< \omega})$ (as in Definition 2.17(2)). Let (\mathcal{T}, \leq) be an arbitrary tree. By compactness we can then find $A' := (a'_\eta)_{\eta \in \mathcal{T}}$ with a'_η tuples in \mathbb{M} such that: $\pi_B(x) := \{\varphi(x, a'_\eta) : \eta \in B\}$ is consistent for every branch $B \subseteq \mathcal{T}$; and for any incomparable nodes $\eta \perp \nu$ in \mathcal{T} , $\{\varphi(x, a'_\eta), \varphi(x, a'_\nu)\}$ is inconsistent (using that every finite induced subtree of \mathcal{T} is isomorphic to an induced subtree of $\omega^{< \omega}$, and that A satisfied these conditions). Note that $\{\pi_B(x) : B \subseteq \mathcal{T} \text{ a branch}\}$ is a family of pairwise inconsistent partial types each of size at most the height of \mathcal{T} , all over the set of parameters A' with $|A'|$ at most the size of \mathcal{T} . Hence $f_T(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}}$ (using Remark 10.8(1)).

(2) As in (1), by multi-TP₁ and compactness, given any tree (\mathcal{T}, \leq) we can find trees of parameters $(a'_\eta : \eta \in \mathcal{T})$ in \mathbb{M} for $\alpha \in \omega$ so that:

- for any choice of branches $B_\alpha \subseteq \mathcal{T}, \alpha \in \omega$, the partial type $\bigcup_{\alpha \in \omega} \{\varphi_\alpha(x, a'_\eta) : \eta \in B_\alpha\}$ is consistent;
- for every $\alpha \in \omega$ and $\eta \perp \nu$ in \mathcal{T} , $\{\varphi_\alpha(x, a'_\eta), \varphi_\alpha(x, a'_\nu)\}$ is inconsistent.

Hence, if the tree \mathcal{T} has height $\leq \kappa$, size $\leq \lambda$ and $\geq \nu$ branches, this gives a set of parameters $A := \bigcup \{a'_\eta : \alpha \in \omega, \eta \in \mathcal{T}\}$ with $|A| \leq \aleph_0 \cdot \lambda = \lambda$ with $\geq \nu^{\aleph_0}$ pairwise-inconsistent partial types over it, each of size $\leq \aleph_0 \cdot \kappa = \kappa$.

Let $\aleph_0 \leq \kappa \leq \lambda$ be arbitrary. Assume first that $\text{cf}(\lambda^{\kappa, \text{tr}}) > \aleph_0$. Then $(\lambda^{\kappa, \text{tr}})^{\aleph_0} = \lambda^{\kappa, \text{tr}} \cdot \sup\{\nu^{\aleph_0} : \nu < \lambda^{\kappa, \text{tr}} \text{ cardinal}\}$ (see e.g. [HSW09, Lemma 1.6.15(c)]). It follows

(using Remark 10.8(1)) that for every $\nu < (\lambda^{\kappa, \text{tr}})^{\aleph_0}$, there is a tree \mathcal{T} of height $\leq \kappa$ and size $\leq \lambda$ with $\geq \nu_0$ branches so that $\nu_0^{\aleph_0} \geq \nu$. Then, by the first paragraph, $f_T^1(\kappa, \lambda) \geq \nu_0^{\aleph_0} \geq \nu$. Hence $f_T^1(\kappa, \lambda) \geq \nu_0^{\aleph_0} \geq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$.

Otherwise $\text{cf}(\lambda^{\kappa, \text{tr}}) \leq \aleph_0$. In this case we can find a tree \mathcal{T} of height $\leq \kappa$ and size $\leq \lambda$ with $\geq \lambda^{\kappa, \text{tr}}$ branches (i.e. the supremum in the definition of $\lambda^{\kappa, \text{tr}}$ is achieved). Namely, we can write $\lambda^{\kappa, \text{tr}} = \sup_{i \in \omega} \nu_i$ for some cardinals $\nu_i < \lambda^{\kappa, \text{tr}}$. By definition of $\lambda^{\kappa, \text{tr}}$, for each $i \in \omega$ there is a tree \mathcal{T}_i of height $\leq \kappa$ and size $\leq \lambda$ with $\geq \nu_i$ branches. We define the tree \mathcal{T} by adding a new root r and, for each $i < \omega$, placing the tree \mathcal{T}_i so that its root r_i is one of the \aleph_0 -many immediate descendants of r . Then the size $|\mathcal{T}|$ of \mathcal{T} is $1 + \sum_{i < \omega} |\mathcal{T}_i| \leq 1 + \aleph_0 \cdot \lambda = \lambda$, the height of \mathcal{T} is at most $1 + \kappa = \kappa$ and the number of branches of \mathcal{T} is $\geq \sum_{i < \omega} \nu_i = \lambda^{\kappa, \text{tr}}$. By the first paragraph again, we thus get $f_T^1(\kappa, \lambda) \geq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$. \square

Using these observations, we can significantly narrow down the possibilities for $f_T(\kappa, \lambda)$:

Proposition 10.15. *Let T be a countable theory with infinite models.*

- (1) T is ω -stable $\Rightarrow f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (2) T is superstable, not ω -stable $\Rightarrow f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda + 2^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (3) T is stable, not superstable $\Rightarrow f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (4) T is supersimple, unstable $\Rightarrow f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda + 2^\kappa$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (5) T is simple, not supersimple, unstable $\Rightarrow f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda^{\aleph_0} + 2^\kappa$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (6) T is not simple $\Rightarrow f_T^1(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}}$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (7) T is not simple, not NIP $\Rightarrow f_T^1(\kappa, \lambda) = f_T(\kappa, \lambda) = \lambda^\kappa$ for all $\lambda \geq \kappa \geq \aleph_0$.
- (8) T is NIP, not stable (= not simple), not multi-order \Rightarrow
 - (a) $f_T^1(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}}$ for all $\lambda \geq \kappa \geq \aleph_0$;
 - (b) $f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda^\kappa$ for all $\lambda^\kappa > 2^\kappa$;
 - (c) $f_T(\kappa, \lambda) \leq \lambda^{\kappa, \text{tr}} = \text{ded } \kappa$ for all $\kappa = \lambda$.
- (9) T is NIP, multi-order \Rightarrow
 - (a) $f_T^1(\kappa, \lambda) \geq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$;
 - (b) $f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda) = \lambda^\kappa$ for all $\lambda^\kappa > 2^\kappa$;
 - (c) $f_T(\kappa, \lambda) \leq (\lambda^{\kappa, \text{tr}})^{\aleph_0} = (\text{ded } \kappa)^{\aleph_0}$ for all $\kappa = \lambda$.

Proof. (1) By Remark 10.4 we have $f_T^1(\kappa, \lambda) \geq \lambda$ for all $\aleph_0 \leq \kappa \leq \lambda$. And if T is ω -stable we have $f_T(\kappa, \lambda) \leq f_T(\lambda, \lambda) \leq \lambda$ by Fact 10.2(1).

(2) By Fact 10.2(2) we have $f_T(\kappa, \lambda) \leq f_T(\lambda) \leq \lambda + 2^{\aleph_0}$, and $f_T^1(\aleph_0, \aleph_0) = 2^{\aleph_0}$. Hence, by Remark 10.4, $f_T^1(\kappa, \lambda) \geq \lambda + 2^{\aleph_0}$ for all $\aleph_0 \leq \kappa \leq \lambda$.

(3) Again we have $f_T(\kappa, \lambda) \leq f_T(\lambda) \leq \lambda^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$ by Fact 10.2(3). On the other hand, assume that T is not superstable. Then it is not supersimple, so by Fact 10.5(2)(c) we have $f_T^1(\kappa, \lambda) \geq \lambda^{\aleph_0}$ for all $\aleph_0 \leq \kappa \leq \lambda$. See also [Cas99, Proposition 4.2] in relation to (1), (2) and (3).

(4) As T is unstable and (super) simple (so NSOP), by Shelah's theorem it has IP. Then $f_T^1(\kappa, \kappa) \geq 2^\kappa$ for all $\kappa \geq \aleph_0$, by Fact 10.2(6) (and $f_T^1(\kappa, \lambda) \geq \lambda$ by Remark 10.4). On the other hand, by Fact 10.5(2)(b) we have that

$f_T(\kappa, \lambda) \leq \lambda + 2^\kappa$ for all κ, λ . By Remark 10.4(2),(3) we conclude that $f_T(\kappa, \lambda) = \lambda + 2^\kappa$ for all $\aleph_0 \leq \kappa \leq \lambda$.

- (5) Again $f_T^1(\kappa, \kappa) \geq 2^\kappa$ for all $\kappa \geq \aleph_0$ as T has IP. On the other hand, as T is simple, by Fact 10.5(1)(b) we have $f_T(\kappa, \lambda) \leq \lambda^{\aleph_0} + 2^\kappa$ for all κ, λ . As T is not supersimple then again by Fact 10.5(2)(c) we have $f_T^1(\kappa, \lambda) \geq \lambda^{\aleph_0}$ for all $\aleph_0 \leq \kappa \leq \lambda$.
- (6) Let T be not simple, and $\aleph_0 \leq \kappa \leq \lambda$ arbitrary. Assume first that T has TP₂, then there is a formula $\varphi(x, y)$ with $|x| = 1$ that has 2-TP₂ (Fact 2.19). By compactness we can then find an array $A := (a_{i,j} : (i, j) \in \kappa \times \lambda)$ so that $\{\varphi(x, a_{i,j}) : j \in \lambda\}$ is 2-inconsistent for each $i \in \kappa$, and $\pi_f(x) := \{\varphi(x, a_{i,f(i)}) : i \in \kappa\}$ is consistent for every $f \in \lambda^\kappa$. Then $\{\pi_f(x) : f \in \lambda^\kappa\}$ is a family a pairwise inconsistent partial types each of size κ , all over the set of parameters A with $|A| \leq \lambda$, so $f_T(\kappa, \lambda) \geq \lambda^\kappa$.

Otherwise, by Fact 2.18, T has TP₁, so we get $f_T^1(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}}$ by Proposition 10.14(1).

- (7) Let $\lambda \geq \kappa \geq \aleph_0$ be arbitrary. If $\lambda^\kappa > 2^\kappa$, then by Proposition 10.9 we have $\lambda^\kappa = \lambda^{\kappa, \text{tr}}$, and as T is not simple by (6) we get $f_T^1(\kappa, \lambda) \geq \lambda^{\kappa, \text{tr}} = \lambda^\kappa$. So assume that $\lambda^\kappa \leq 2^\kappa$. As T has IP, we have that $f_T^1(\kappa, \lambda) \geq f_T^1(\kappa, \kappa) \geq 2^\kappa \geq \lambda^\kappa$ by Fact 10.2(6).
- (8) As T is NIP unstable, it is also not simple, hence $f_T^1(\kappa, \kappa) \geq \lambda^{\kappa, \text{tr}}$ by (6). Then (b) holds by Proposition 10.9. And (c) holds by Fact 10.2(5) (and Remark 10.8(3)).
- (9) As T is NIP and multi-order, by Proposition 10.13, T is multi-TP₁ witnessed by some formulas $(\varphi_i(x, y_i))_{i \in \omega}$ with $|x| = 1$. Then, by Proposition 10.14(2), $f_T^1(\kappa, \lambda) \geq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$ for all $\aleph_0 \leq \kappa \leq \lambda$. Again, (b) holds by Proposition 10.9. And (c) holds by Fact 10.2(4). □

By Proposition 10.15, Conjecture 10.6(1) follows from the following (in which case the explicit list of possible functions $f_T(\kappa, \lambda)$ is given in Proposition 10.15):

Conjecture 10.16. *Let T be a countable theory and $\aleph_0 \leq \kappa \leq \lambda$.*

- (1) *If T is NIP, then $f_T(\kappa, \lambda) \leq (\lambda^{\kappa, \text{tr}})^{\aleph_0}$.*
 (2) *If T is not multi-order, then $f_T(\kappa, \lambda) \leq \lambda^{\kappa, \text{tr}}$.*

(Without loss of generality we may restrict to the case $\lambda^\kappa \leq 2^\kappa$ — as for $\lambda^\kappa > 2^\kappa$ this holds trivially by Proposition 10.9.)

Corollary 10.17. *Conjecture 10.16, and hence Conjecture 10.6(1), holds assuming GCH.*

Proof. For any infinite $\kappa \leq \lambda$ with $\lambda^\kappa \leq 2^\kappa$ we have $\kappa < \text{ded } \kappa = \kappa^{\kappa, \text{tr}} \leq \lambda^{\kappa, \text{tr}} \leq \lambda^\kappa \leq 2^\kappa$. Hence, assuming GCH, $\lambda^\kappa = \lambda^{\kappa, \text{tr}}$, so the bound in Conjecture 10.16 holds trivially. □

Corollary 10.18. *Conjecture 10.6(2) is false.*

Proof. By Proposition 10.15, if a countable theory T is neither simple, nor NIP, then the function $f_T(\kappa, \lambda) = \lambda^\kappa$ is maximal for all infinite $\kappa \leq \lambda$. But such a T can be taken to have NTP₂ or not. □

We also ask if $f_T(\kappa, \lambda) = f_T^1(\kappa, \lambda)$ for all T, κ, λ (unlike that case of complete types and $f_T(\kappa)$, this is not automatic for counting partial types; note that Conjecture 10.16 combined with Proposition 10.15 implies that this holds for countable T). We also mention that a different two-cardinal invariant of first-order theories generalizing $f_T(\kappa)$, namely a function of two cardinals κ and λ giving the supremum of the possible number of types over a model of size λ that do not fork over a sub-model of size κ , is considered in [CKS16].

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