Abstracts

Model-theoretic Elekes-Szabó

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1. Introduction

Erdős and Szemerédi [1] observed the following sum-product phenomenon: there is some $c \in \mathbb{R}_{>0}$ such that for any finite set $A \subseteq \mathbb{R}$, $\max\{|A+A|, |A \cdot A|\} \geq |A|^{1+c}$. Later, Elekes and Rónyai [2] generalized this by showing that for any polynomial $f(x,y)$ we must have $|f(A \times A)| \geq |A|^{1+c}$, unless f is either additive or multiplicative (i.e. of the form $g(h(x)+i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials g, h, i respectively). Elekes and Szabó [3] provide a conceptual generalization, showing that for any irreducible polynomial $F(x,y,z)$ depending on all of its coordinates such that its set zero set has dimension 2, if F has a maximal possible number of zeroes n^2 on finite $n \times n \times n$ grids, then F is in a finite-to-finite correspondence with the graph of multiplication of an algebraic group (in the special case above, either the additive or the multiplicative group of the field). Recently, several generalizations were obtained for relations of higher dimension and arity that we review in the next section. Here we announce a generalization of this result to hypergraphs of any arity and dimension definable in a large class of stable structures which includes differentially closed fields and compact complex manifolds, as well as for arbitrary o-minimal structures (to appear in [4]).

2. Elekes-Szabó principle

We fix a structure $\mathcal{M}$, definable sets $X_1, \ldots, X_s$, and a definable relation $Q \subseteq \bar{X} = X_1 \times \ldots \times X_s$. We write $A_i \subseteq_n X_i$ if $A_i \subseteq X_i$ with $|A_i| \leq n$. By a *grid on* $\bar{X}$ we mean a set $\bar{A} \subseteq \bar{X}$ with $\bar{A} = A_1 \times \ldots \times A_s$ and $A_i \subseteq X_i$. By an *n-grid on* $\bar{X}$ we mean a grid $\bar{A} = A_1 \times \ldots \times A_s$ with $A_i \subseteq_n X_i$.

2.1. **Fiber-algebraic relations.** A relation $Q \subseteq \bar{X}$ is *fiber-algebraic* if there is some $d \in \mathbb{N}$ such that for any $1 \leq i \leq s$ we have

$$\mathcal{M} \models \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_s \exists^{\leq d} x_i \, Q(x_1, \ldots, x_s).$$

For example, if $Q \subseteq X_1 \times X_2 \times X_3$ is fiber-algebraic, then for any $A_i \subseteq_n X_i$ we have $|Q \cap A_1 \times A_2 \times A_3| = dn^2$. Conversely, let $Q \subseteq \mathbb{C}^3$ be given by $x_1 + x_2 - x_3 = 0$, and let $A_1 = A_2 = A_3 = \{0, \ldots, n-1\}$. Then $|Q \cap A_1 \times A_2 \times A_3| = \frac{n(n+1)}{2} = \Omega\left(n^2\right)$. This indicates that the upper and lower bounds match for the graph of addition in an abelian group (up to a constant) — and the Elekes-Szabó principle suggests that in many situations this is the only possibility. Before making this precise, we introduce some notation.

2.2. **Grids in general position.** From now on we will assume that $\mathcal{M}$ is equipped with some notion of integer-valued dimension on definable sets, to be specified later. A good example to keep in mind is Zariski dimension on constructible subsets of $\mathcal{M} := (\mathbb{C}, +, \times) \models \mathrm{ACF}_0$, the theory of algebraically closed fields of characteristic 0.

Let X be an $\mathcal{M}$-definable set and let $\mathcal{F}$ be a (uniformly) $\mathcal{M}$-definable family of subset of X. For $l \in \mathbb{N}$ we say that a set $A \subseteq X$ is *in* $(\mathcal{F}, l)$*-general position* if $|A \cap F| \leq l$ for every $F \in \mathcal{F}$ with $\dim(F) < \dim(X)$.

Let X_i, $i = 1, \ldots, s$, be $\mathcal{M}$-definable sets. Let $\bar{\mathcal{F}} = (\mathcal{F}_1, \ldots, \mathcal{F}_s)$, where $\mathcal{F}_i$ is a definable family of subsets of X_i. For $l \in \mathbb{N}$ we say that a grid $\bar{A}$ on $\bar{X}$ is in $(\bar{\mathcal{F}}, l)$*-general position* if each A_i is in $(\mathcal{F}_i, l)$-general position.

For example, if X is strongly minimal and $\mathcal{F}$ is any definable family of subsets of X, then for any large enough $l = l(\mathcal{F}) \in \mathbb{N}$, every $A \subseteq X$ is in $(\mathcal{F}, l)$-general position. On the other hand, let $X = \mathbb{C}^2$ and let $\mathcal{F}_d$ be the family of algebraic curves of degree d. If $l < d$, then any set $A \subseteq X$ is not in $(\mathcal{F}_d, l)$-general position.

2.3. **Generic correspondence with group multiplication.** Let $Q \subseteq \bar{X}$ be a definable relation and $(G, \cdot)$ a type-definable group in $\mathbb{M}^{\mathrm{eq}}$ which is connected (i.e. $G = G^0$). We say that Q is in a *generic correspondence with multiplication in* G if there exist elements $g_1, \ldots, g_s \in G(\mathbb{M})$, where $\mathbb{M}$ is a saturated elementary extension of $\mathcal{M}$, such that:

(1) $g_1 \cdot \ldots \cdot g_s = e$;
(2) $g_1, \ldots, g_{s-1}$ are independent generics in G over $\mathcal{M}$, i.e. each g_i doesn't belong to any definable set of dimension smaller than G definable over $\mathcal{M} \cup \{g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{s-1}\}$;
(3) For each $i = 1, \ldots, s$ there is a generic element $a_i \in X_i$ interalgebraic with g_i over $\mathcal{M}$, such that $\models Q(a_1, \ldots, a_s)$.

If X_i are irreducible, then (3) holds for all $g_1, \ldots, g_s \in G$ satisfying (1) and (2), providing a generic finite-to-finite correspondence between Q and the graph of $(s-1)$-fold multiplication in G.

2.4. **The Elekes-Szabó principle.** Let $X_1, \ldots, X_s$ be definable sets in $\mathcal{M}$ with $\dim(X_i) = k$ and X_i irreducible (i.e. can't be split into two disjoint definable sets of full dimension). We say that they satisfy the *Elekes-Szabó principle* if for any irreducible fiber-algebraic definable relation $Q \subseteq \bar{X}$, one of the following holds:

(1) Q admits power saving: there exist some $\varepsilon \in \mathbb{R}_{>0}$ and some definable families $\mathcal{F}_i$ on X_i such that: for any $l \in \mathbb{N}$ and any n-grid $\bar{A} \subseteq \bar{X}$ in $(\bar{F}, l)$-general position, we have $|Q \cap \bar{A}| = O_l(n^{(s-1)-\varepsilon})$;
(2) Q is in a generic correspondence with multiplication in a type-definable *abelian* group of dimension k.

Below are the previously known cases of the Elekes-Szabó principle:

- [3] $\mathcal{M} \models \mathrm{ACF}_0$, $s = 3$, k arbitrary;
- [5] $\mathcal{M} \models \mathrm{ACF}_0$, $s = 4$, $k = 1$;

- [6] $\mathcal{M} \models \mathrm{ACF}_0$, s and k arbitrary, recongized that the arising groups are abelian (they work with a more relaxed notion of general position and arbitrary codimension, however no bounds on ε);
- [7] $\mathcal{M}$ is any strongly minimal structure interpretable in a distal structure, $s = 3$, $k = 1$.

Theorem 1. [4] *The Elekes-Szabó principle holds in the following two cases:*

(1) *$\mathcal{M}$ is a stable structure interpretable in a distal structure, with respect to $\mathfrak{p}$-dimension (see below).*
(2) *$\mathcal{M}$ is an o-minimal structure, with respect to the topological dimension (in this case, on a type-definable generic subset of $\bar{X}$, we get a definable coordinate-wise* bijection *of Q with the graph of multiplication of G).*

Here we only discuss the stable case (1). Examples of structures satisfying the assumption of (1) are models of ACF_0, $\mathrm{DCF}_{0,m}$ (i.e. differentially closed fields with m commuting derivations), CCM (the theory of compact complex manifolds). Our method provides explicit bounds on ε for power saving in these cases.

3. Ingredients of the proof in the stable case

3.1. **$\mathfrak{p}$-dimension.** We choose a saturated elementary extension $\mathbb{M}$ of a *stable* structure $\mathcal{M}$. By *a $\mathfrak{p}$-pair* we mean a pair $(X, \mathfrak{p}_X)$, where X is an $\mathcal{M}$-definable set and $\mathfrak{p}_X \in S(\mathcal{M})$ is a complete stationary type on X. Assume we are given $\mathfrak{p}$-pairs $(X_i, \mathfrak{p}_i)$ for $1 \leq i \leq s$. We say that a definable $Y \subseteq X_1 \times \ldots \times X_s$ is $\mathfrak{p}$-generic if $Y \in \mathfrak{p}_1 \otimes \ldots \otimes \mathfrak{p}_s|_{\mathbb{M}}$. Finally, we define the $\mathfrak{p}$-dimension via $\dim_{\mathfrak{p}}(Y) \geq k$ if for some projection π of $\bar{X}$ onto k components, $\pi(Y)$ is $\mathfrak{p}$-generic. This $\mathfrak{p}$-dimension enjoys definability and additivity properties crucial for our arguments that may fail for Morley rank in general ω-stable theories such as DCF_0. However, if X is a definable subset of finite Morley rank k and degree one, taking $\mathfrak{p}_X$ to be the unique type on X of Morley rank k, we have that $k \cdot \dim_{\mathfrak{p}} = \mathrm{MR}$, and Theorem 1(1) implies that the Elekes-Szabó principle holds with respect to Morley rank in this case.

3.2. **Distality and incidence bounds.** Distal structures were introduced in [8], and connections with combinatorics and generalized incidence bounds were established in [9, 10, 11]. The key result for us is the following generalized "Szemerédi-Trotter" theorem:

Theorem 2. [11, 4] *If $E \subseteq U \times V$ is a binary relation definable in a distal structure and E is $K_{s,2}$-free for some $s \in \mathbb{N}$, then there is some $\delta > 0$ such that: for all $A \subseteq_n U, B \subseteq_n V$ we have $|E \cap A \times B| = O(n^{\frac{3}{2}-\delta})$.*

3.3. **Recovering groups from abelian m-gons.** Working in a stable theory, an *m-gon* over A is a tuple $a_1, \ldots, a_m$ such that any $m-1$ of its elements are independent over A, and any element in it is in the algebraic closure of the other ones and A. We say that an m-gon is *abelian* if, after any permutation of its elements, we have $a_1 a_2 \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}_A(a_1 a_2) \cap \mathrm{acl}_A(a_3 \ldots a_m)} a_3 \ldots a_m$.

If $(G, \cdot)$ is a type-definable abelian group, $g_1, \ldots, g_{m-1}$ are independent generics in G and $g_m := g_1 \cdot \ldots \cdot g_{m-1}$, then $g_1, \ldots, g_m$ is an abelian m-gon (associated to G). Conversely,

Theorem 3. [4] *Let $a_1, \ldots, a_m$ be an abelian m-gon. Then there is a type-definable (in $\mathcal{M}^{eq}$) connected abelian group $(G, \cdot)$ and an abelian m-gon $g_1, \ldots, g_m$ associated to G, such that after a base change each g_i is inter-algebraic with a_i.*

An analogous result was obtained independently by Hrushovski.

3.4. **Distinction of cases in Theorem 1.** We may assume $\dim(Q) = s - 1$, and let $\bar{a} = (a_1, \ldots, a_s)$ be a generic tuple in Q over $\mathcal{M}$. As Q is fiber-algebraic, $\bar{a}$ is an s-gon over $\mathcal{M}$.

Theorem 4. [4] *One of the following is true:*

(1) *For $u = (a_1, a_2)$ and $v = (a_3, \ldots, a_s)$ we have $u \perp_{\mathrm{acl}_M(u) \cap \mathrm{acl}_M(v)} v$.*
(2) *Q, as a relation on $U \times V$, for $U = X_1 \times X_2$ and $V = X_3 \times \ldots \times X_s$, is a "pseudo plane".*

In case (2) the incidence bound from Theorem 2 can be applied inductively to obtain power saving for Q. Thus we may assume that that for any permutation of $\{1, \ldots, s\}$ we have

$$a_1 a_2 \perp_{\mathrm{acl}_M(a_1 a_2) \cap \mathrm{acl}_M(a_3 \ldots a_s)} a_3 \ldots a_s,$$

i.e. the s-gon $\bar{a}$ is abelian, and Theorem 3 can be applied to establish generic correspondence with a type-definable abelian group.

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