Abstracts

Towards higher classification theory

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1. Introduction

Model theory provides, among other things, methods of converting asymptotic quantitative questions about properties of finite hypergraphs into qualitative questions about the "shape", "volume" or "dimension" of certain limiting infinite objects to which the infinitary model-theoretic machinery can be applied. Shelah's classification program [1] isolates several combinatorial dividing lines (stability, NIP, distality, etc.) separating mathematical structures exhibiting various degrees of wild, or Gödelian, behavior, from the tame ones in which one develops a "geometric" theory akin to algebraic or semi-algebraic geometry for definable sets in such structures. These dividing lines are amazingly robust, and have been rediscovered in various branches of mathematics.

These tameness notions in Shelah's classification theory are typically given by restrictions on the combinatorial complexity of definable binary relations, by forbidding certain induced subgraphs (e.g. T is *stable* if no definable binary relation can contain arbitrary large finite half-graphs; and *NIP* if sufficiently large random bipartite graphs are omitted). A typical result then demonstrates that binary relations are "approximated" by the unary ones in some form, up to a "small" error. For example, stationarity of forking in stable theories says that given $p(x), q(y)$ types over a model M, there exists a *unique type* $r(x, y)$ over M so that if $(a, b) \models r$ then $a \models p, b \models q$ and $a \mathop{\smile\!\!\!\!\!\!|}_M b$ — that is, there is a unique type $r(x, y)$ extending $p(x) \cup q(y)$, up to the forking formulas $\varphi(x, y) \in \mathcal{L}(M)$. Another example: T is *distal* if and only if for any $p(x), q(y)$ global invariant types that commute, there is a unique global type $r(x, y)$ extending $p(x) \cup q(y)$.

Recently a number of results began to emerge concerning the higher arity generalizations of these phenomena, both in the context of pure model theory and in connection to hypergraph combinatorics: under some restricting assumption on the definable relations of arity $n+1$, demonstrate an "approximation" by relations each involving at most n out of $n+1$ variables, up to a "small error". Mirroring the passage from graphs to hypergraphs in combinatorics, this leads to significant growth in complexity of the occurring phenomena. We overview some of these developments focusing on of n-dependent theories (with the case $n = 1$ corresponding to NIP) introduced by Shelah, n-stability (several possible definitions have recently emerged in the literature, but very much remain to be explored), n-distality (recently introduced by Walker), and connections to higher amalgamation and stationarity, as well as implications for the algebraic structures definable in such theories.

2. N-dependence

A higher order generalization of NIP, the class of *k-dependent theories*, was introduced by Shelah in [2], with the 1-dependent case corresponding to the class of NIP theories, and basic properties of k-dependent theories were investigated in [3], in particular making an explicit definition of the VC_k-dimension.

We fix a complete theory T in a language $\mathcal{L}$. For $k \geq 1$, a formula $\varphi(x; y_1, \ldots, y_k)$ is *k-dependent* if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of T such that $A = \prod_{i=1}^{n} A_i$ is shattered by φ: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $\mathcal{M} \models \varphi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$. T is *k-dependent* if all formulas are k-dependent. T is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent. We have: 1-dependent = NIP $\subsetneq$ 2-dependent $\subsetneq \ldots$, with all inclusions strict as witnessed e.g. by the theory of the generic k-hypergraph.

In some sense all currently known "algebraic" examples of k-dependent theories are built from multilinear forms over NIP fields. By Cherlin-Hrushovski, smoothly approximable structures are 2-dependent, and coordinatizable via bilinear forms over finite fields. Infinite extra-special p-groups are strictly 2-dependent [4], and strictly k-dependent pure groups constructed using Mekler's construction [5] are essentially of this form as well, using Baudisch's interpretation in alternating bilinear maps. More generally:

Theorem 1. *([6] for $k = 2$, [7] in general) If the field K is NIP, then the theory T of alternating n-linear forms over K (with sorts for the field and for the vector space, generalizing Granger) is (strictly) n-dependent.*

This leads one to speculate that if T is k-dependent, then it is "linear, or 1-based" relative to its NIP part. One precise version of this conjecture is:

Conjecture 1. *If K is an k-dependent field (pure, or with valuation, derivation, etc.), then K is NIP.*

There is some mounting evidence for this conjecture: k-dependent fields are Artin-Schreier closed ([4], generalizing Kaplan-Scanlon-Wagner for $k = 1$), valued fields of positive characteristic are Henselian ([6], generalizing Johnson for $k = 1$), the question for valued fields reduces to pure fields (Boissonneau). A key general result used in the proof of Theorem 1 is:

Theorem 2 (Composition Lemma)**.** *Let $\mathcal{M}$ be an $\mathcal{L}'$-structure such that its reduct to a language $\mathcal{L} \subseteq \mathcal{L}'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \ldots, x_d)$ be an $\mathcal{L}$-formula, and $(y_0, \ldots, y_k)$ be arbitrary $k+1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \ldots, i_k^t \leq k$ be arbitrary, and let $f_t : M_{y_{i_1^t}} \times \ldots \times M_{y_{i_k^t}} \to M_{x_t}$ be an arbitrary $\mathcal{L}'$-definable k-ary function. Then the formula*

$$\psi(y_0; y_1, \ldots, y_k) := \varphi\left(f_1(y_{i_1^1}, \ldots, y_{i_k^1}), \ldots, f_d(y_{i_1^d}, \ldots, y_{i_k^d})\right)$$

is k-dependent.

The following is a characterization of k-dependence in terms of a "hypergraph regularity lemma", generalizing the $k = 1$ case from [9]:

Theorem 3. *(C., Towsner [8]) Assume that T is k-dependent, $k' \geq k+1$, $\mathbb{M} \models T$ and let $\mu_1, \ldots, \mu_{k'}$ be global Keisler measures on the definable subsets of the sorts $\mathbb{M}^{x_1}, \ldots, \mathbb{M}^{x_{k'}}$ respectively, such that each μ_i is Borel-definable and all these measures commute, i.e. $\mu_i \otimes \mu_j$ for all $i, j \in [k']$. Then for every formula $\varphi(x_1, \ldots, x_{k'}) \in \mathcal{L}(\mathbb{M})$ and $\varepsilon \in \mathbb{R}_{>0}$ there exist some formula $\psi(x_1, \ldots, x_{k'})$ which is a Boolean combination of finitely many $(\leq k)$-ary formulas each given by an instances of φ with some parameters placed in all but at most k variables, so that taking $\mu := \mu_1 \otimes \ldots \otimes \mu_{k'}$ we have $\mu(\varphi \triangle \psi) < \varepsilon$.*

It is also proved in that paper that if T is a k-dependent first-order theory (classical or continuous), then its Keisler randomization T^R is also k-dependent, generalizing Ben Yaacov for $k = 1$.

3. N-distality

Definition 1. *(Walker [10]) A theory is* n-distal *if it satisfies the following condition. Assume that $(a_i : i \in I)$ is an indiscernible sequence indexed by a dense linear order I, $I = I_0 + I_1 + \ldots + I_{n+1}$ with each I_j dense without endpoints, and $b_1, \ldots, b_{n+1}$ are so that: for any $0 \leq t \leq n$, we have that the sequence $I_0 + b_0 + \ldots + I_{t-1} + b_{t-1} + I_t + I_{t+1} + b_{t+1} + \ldots + I_n + b_n + I_{n+1}$ is indiscernible (i.e. we are omitting b_t here). Then the sequence $I_0 + b_0 + \ldots + b_n + I_{n+1}$ is indiscernible (with all $b_t, 0 \leq t \leq n$ placed in the corresponding cuts).*

The following generalizes a standard characterization of distality:

Fact 1. [10] *If T is n-distal, then for any global invariant types $p_i(x_i), 0 \leq i \leq n$ that are* pairwise commuting, *we have $\bigcup_{0 \leq t \leq n} \bigotimes_{0 \leq i \leq n, i \neq t} p_i(x_i) \vdash \bigotimes_{0 \leq i \leq n} p_i(x_i)$. That is, the type $\bigotimes_{0 \leq i \leq n} p_i$ in $n+1$ variables is determined by all of its restrictions to n variables.*

Turns out that n-distality is connected to certain notions of triviality of forking (as studied by Poizat and others) between generically stable types (for $k = 1$, in the sense that they are all realize).

Definition 2. *Let T be a stable theory and $k \geq 1$. We say that T is*

1. k-trivial *if for any tuples $(a_i : i < k+2)$ and a set A, if every $k+1$ of the a_i's form an independent set over A (in the sense of forking), then every $\{a_i : i < k+2\}$ is also an independent set over A.*
2. totally k-trivial *if for any tuples a, $(b_i : i < k+1)$ and a set A, if a is independent from any k of the b_i's over A, then it is also independent from all $k+1$ of them over A (note that we are not requiring the b_i's to be independent over A).*
3. *For $k \geq 1$, a theory T is* indiscernibly k-trivial *if for any infinite sequence $\mathcal{I}$ and tuples $(a_t : t < k+1)$, if $\mathcal{I}$ is indiscernible over $(a_t : t \in s)$ for every $s \subseteq \{0, 1, \ldots, k\}$ with $|s| = k$, then $\mathcal{I}$ is indiscernible over $(a_t : t < k+1)$.*

Fact 2. [10] *Let T be a stable theory and $k > 0$. Then T is k-trivial if and only if T is $(k+1)$-distal.*

A theory T is *strongly* 2-*distal* if for any sequence $\mathcal{I}_0+b_0+\mathcal{I}_1$ and tuples a_0, a_1, if $\mathcal{I}_0+\mathcal{I}_1$ is indiscernible over a_0a_1, $\mathcal{I}_0+b_0+\mathcal{I}_1$ is indiscernible over a_0 and $\mathcal{I}_0+b_0+\mathcal{I}_1$ is indiscernible over a_1, then $\mathcal{I}_0+b_0+\mathcal{I}_1$ is indiscernible over a_0a_1. We observe:

Theorem 4. *If T is stable, then the following are equivalent:*

(1) *T is strongly* 2*-distal,*
(2) *T is indiscernibly trivial,*
(3) *T is totally trivial.*

Whether triviality is equivalent to k-triviality (equivalently, k-distality implies 2-distality) in stable theories is an old question of Poizat (known to hold in superstable theories). Theorem 4 combined with Poizat's examples answers Walker's question [10]: there exist stable 2-distal not strongly 2-distal theories.

4. Connections to higher amalgamation and stationarity

Higher amalgamation was studied by a number of authors, starting with Shelah's work on stability in AEC's, Hrushovski in the study of the saturation spectrum and of generalized imaginaries, continued in a series of papers by Goodrick, Kim, Kolesnikov and others.

Definition 3. *For $n \in \omega$, let $[n] = \{1, \dots, n\} \in \omega$. For a set X, we let $\mathcal{P}(X)$ be the set of all subsets of X, $\mathcal{P}_{<n}(X)$ ($\mathcal{P}_{\leq n}(X)$) the set of all subsets of X of size less (respectively, less or equal) than n, and $\mathcal{P}^-(X) := \mathcal{P}(X) \setminus \{X\}$.*

We let T be a complete *simple* first-order theory in a language $\mathcal{L}$, and we work in $\mathbb{M}^{\text{heq}}$, the expansion of $\mathbb{M}$ by the hyper-imaginaries. As usual, $\downarrow$ denotes forking independence and $\text{bdd}(A)$ is the bounded closure of the set A in $\mathbb{M}^{\text{heq}}$.

Definition 4. *Let X be an arbitrary small set, and $S \subseteq \mathcal{P}(X)$ be non-empty and closed under subsets (so in particular $\emptyset \in S$). Let $\{r_s(x_s) : s \in S\}$ be a family of complete types over $\emptyset$ (where each x_s is a possibly infinite tuple of variables). We say that such a family of types is* independent *if:*

(1) *if $a_\emptyset \models r_\emptyset$, then the set of elements of the tuple $a_\emptyset$ is boundedly closed;*
(2) *if $s, t \in S$ and $s \subset t$, then $x_s \subset x_t$ and $r_s \subset r_t$;*
(3) *for all $s, t \in S$ we have $x_s \cap x_t = x_{s\cap t}$;*
(4) *if $s \in S$ and $a_s \models r_s$, then:*
 (a) *the set $\left\{a_{\{t\}} : t \in S\right\}$ is independent over $a_\emptyset$, where $a_{\{t\}}$ is a subtuple of a_s corresponding to the subtuple of the variables $x_{\{t\}} \subseteq x_s$;*
 (b) *the set of elements of the tuple a_s is equal to $\text{bdd}\left(\bigcup_{t\in S} a_{\{t\}}\right)$, and the map $a_s \to x_s$ between the realizations and the variables is a bijection.*

Definition 5. (1) *For $n \geq 1$, T satisfies (*independent*) n-*amalgamation *if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists a complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.*

(2) *T satisfies (*independent*) n-*uniqueness *if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists at most one complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.*
(3) *T satisfies n-amalgamation (n-uniqueness)* over a set $A \subseteq \mathbb{M}$ *if (1) (respectively, (2)) holds for every independent system of types with $r_\emptyset =$* tp(bdd(A)).
(4) *T satisfies* complete n-amalgamation *(or $\leq n$-*amalgamation*) if T satisfies m-amalgamation for all $1 \leq m \leq n$.*

Theorem 5. *Given $n \geq 1$, let T be a simple theory with $\leq (n+2)$-amalgamation. Then T is n-dependent if and only if T has $(n+1)$-uniqueness (over models).*

For $n = 1$ this corresponds to the well-known fact that if T is simple (hence satisfies $\leq$ 3-amalgamation over models) and there exists a non-stationary type (i.e. 2-stationarity fails), then T is not NIP. Theorem 5 also gives us a collapse of 2-dependence and several notions of 2-stability considered in the literature.

Definition 6 (Takeuchi)**.** *A partitioned formula $\varphi(x; y_1, y_2)$ has OP_2 if there exist sequences $(a_i)_{i\in\omega}, (b_j)_{j\in\omega}$ with $a_i \in \mathbb{M}^{y_1}, b_j \in \mathbb{M}^{y_2}$ so that for every strictly increasing $f : \omega \to \omega$ there exists $c_f \in \mathbb{M}^x$ satisfying $\models \varphi(c_f, a_i, b_j) \iff i \leq f(j)$ for all $(i, j) \in \omega^2$.*

A related property FOP_2 with increasing functions replaced by arbitrary functions $f : \omega \to \omega$ is considered by Terry and Wolf. We let $\mathcal{C}_\prec := (\mathbb{L}, C, \prec)$ be the *generic countable convexly ordered binary branching C-relation*, i.e. the Fraïssé limit of all finite convexly ordered binary branching C-relations. The following property is related to *treeless* theories considered by Kaplan, Ramsey, Simon:

Definition 7. *A theory T is* $\mathcal{C}$-less *if there is no formula $\varphi(x, y, z)$ and $(a_g : g \in \mathbb{L})$ such that $\models \varphi(a_f, a_g, a_h) \iff \mathcal{C}_\prec \models C(f, g, h)$.*

Theorem 6. *The following are equivalent:*

(1) *T is not $\mathcal{C}$-less;*
(2) *there exists a $\mathcal{C}_\prec$-indiscernible which is not $(\mathbb{L}, \prec)$-indiscernible;*
(3) *there exists a $\mathcal{C}_\prec$-indiscernible $(a_g : g \in \mathbb{L})$ and $\varphi(x, y, z)$ so that*

$$\models \varphi(a_f, a_g, a_h) \iff \mathcal{C} \models C(f, g, h).$$

It is easy to see that each of $\mathcal{C}$-less, no OP_2 and no FOP_2 imply 2-dependence, and under 4-amalgamation we get a converse:

Theorem 7. *If T is simple with $\leq$ 4-amalgamation, then the following are equivalent:*

(1) *T satisfies* 3-*uniqueness;*
(2) *T is* 2-*dependent;*
(3) *T has no OP_2;*
(4) *T has no FOP_2;*
(5) *T is $\mathcal{C}$-less.*

References

[1] S. Shelah, *Classification theory: and the number of non-isomorphic models*, Elsevier (1990).
[2] S. Shelah. *Strongly dependent theories*, Israel Journal of Mathematics, **204** (2014), 1–83.
[3] A. Chernikov, D. Palacin, K. Takeuchi. *On n-dependence*, Notre Dame J. Form. Log., **60** (2019), 195–214.
[4] N. Hempel. *On ndependent groups and fields*, Mathematical Logic Quarterly, **62** (2016), 215–224.
[5] A. Chernikov, N. Hempel. *Meklers construction and generalized stability*, Israel Journal of Mathematics, **230** (2019), 745–769.
[6] A. Chernikov, N. Hempel. *On n-dependent groups and fields II*, Forum of Mathematics, Sigma, **9** (2021)
[7] A. Chernikov, N. Hempel. *On n-dependent groups and fields III*, in preparation (2023+)
[8] A. Chernikov, H. Towsner. *Hypergraph regularity and higher arity VC-dimension*, Preprint, arXiv:2010.00726 (2020)
[9] A. Chernikov, S. Starchenko. *Definable regularity lemmas for NIP hypergraphs*, The Quarterly Journal of Mathematics, **74** (2021), 1401–1433.
[10] R. Walker. *Distality rank*, The Journal of Symbolic Logic, (2019), 1–32.

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