Mekler's construction and generalized stability

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Mekler's construction

- Let $p > 2$ be prime.
- Let T be any theory in a finite relational language.
- [Mekler'81] A uniform construction of a group $G(\mathcal{M})$ for every $\mathcal{M} \models T$, a theory T^* of all groups $\{G(\mathcal{M}) : \mathcal{M} \models T\}$ and an interpretation Γ of T in T^* s.t.:
  - T^* is a theory of nilpotent groups of class 2 and of exponent p,
  - if $G \models T^*$, then $\exists \mathcal{M} \models T$ s.t. $G(\mathcal{M}) \equiv G$,
  - For $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \equiv \mathcal{N} \iff G(\mathcal{M}) \equiv G(\mathcal{N})$,
  - $\Gamma(G(\mathcal{M})) \cong \mathcal{M}$.
- Idea:
  - Bi-interpret $\mathcal{M}$ with a *nice* graph C.
  - Define a group $G(C)$ generated freely by the vertices of C, imposing that two generators commute $\iff$ they are connected by an edge in C.
- This kind of coding of graphs is known in probabilistic group theory, recursion theory, etc.

What model-theoretic properties are preserved?

- This is not a bi-interpretation (e.g., the resulting group is never ω-categorical), however some model-theoretic tameness properties are known to be preserved.
- [Mekler '81] For any cardinal κ, $\mathrm{Th}(\mathcal{M})$ is κ-stable $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ is κ-stable.
- [Baudisch, Pentzel '02] $\mathrm{Th}(\mathcal{M})$ is simple $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ is simple.
- [Baudisch '02] Assuming stability, $\mathrm{Th}(\mathcal{M})$ is CM-trivial $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ is CM-trivial.
- We investigate what further properties from Shelah's classification are preserved.

k-dependent theories

- We fix a complete theory T in a language $\mathcal{L}$. For $k \geq 1$ we define:

Definition

[Shelah]

- A formula $\phi(x; y_1, \ldots, y_k)$ is *k-dependent* if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of T such that $A = \prod_{i=1}^{n} A_i$ is *shattered* by ϕ, where "A shattered" means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $M \models \phi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$.
- T is *k-dependent* if all formulas are k-dependent.
- T is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent.

- T is 1-dependent $\iff$ T is NIP.
- 1-dependent $\subsetneq$ 2-dependent $\subsetneq \ldots$ as witnessed by e.g. the theory of the random k-hypergraph.

k-dependent fields?

- **Problem.** Are there strictly k-dependent fields, for $k > 1$?
- **Conjecture.** There are no *simple* strictly k-dependent fields, for $k > 1$.
- [Hempel '15] Let K be an infinite field.
    1. If $\mathrm{Th}(K)$ is n-dependent, then K is Artin-Schreier closed.
    2. If K is a PAC field which is not separably closed, then $\mathrm{Th}(K)$ is not k-dependent for any $k \in \omega$.
- (2) is due to Parigot for $k = 1$, and if K is pseudofinite, by Beyarslan K interprets the random k-hypergraph for all $k \in \omega$.

k-dependent groups

- Let T be a theory and G a type-definable group (over $\emptyset$), and $A \subseteq \mathbb{M}$ a small subset.
- Let G_A^{00} be the minimal type-definable over A subgroup of G of bounded index.

Fact

T is NIP $\implies G_A^{00} = G_\emptyset^{00}$ for all small A.

Example

Let $G := \bigoplus_\omega \mathbb{F}_p$. Let $\mathcal{M} := (G, \mathbb{F}_p, 0, +, \cdot)$ with $\cdot$ the bilinear form $(a_i) \cdot (b_i) = \sum_i a_i b_i$ from G to $\mathbb{F}_p$.
Then G is 2-dependent and $G_A^{00} = \{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$ — gets smaller when enlarging A.

Fact

[Shelah] Let T be 2-dependent. Then for a suitable cardinal κ, if $\mathcal{M} \prec \mathbb{M}$ is κ-saturated and $|B| < \kappa$, then $G_{M \cup B}^{00} = G_M^{00} \cap G_{A \cup B}^{00}$ for some $A \subseteq M$, $|A| < \kappa$.

- This can be viewed as a trace of modularity.

Mekler's construction preserves k-dependence

- No examples of strictly k-dependent groups for $k > 2$ were known.

Theorem

[C., Hempel '17] For any $k \in \omega$, $\mathrm{Th}(\mathcal{M})$ k-dependent $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ is k-dependent.

- Applying Mekler's construction to the random k-hypergraph, we get:

Corollary

For every $k \in \omega$, there is a strictly k-dependent pure group G_k (moreover, $\mathrm{Th}(G_k)$ simple by Baudisch).

A proof for NIP, 1

- For a complete theory T, its *stability spectrum* is the function $f_T(\kappa) := \sup\{|S_1(M)| : M \models T, |M| = \kappa\}$.
- $\operatorname{ded}(\kappa) :=$
  $\sup\{|I| : I$ is a linear order with a dense subset of size $\kappa\}$.

Fact

[Shelah] Let the language of T be countable.

1. *If T is* NIP, *then $f_T(\kappa) \leq (\operatorname{ded}\kappa)^{\aleph_0}$ for all infinite cardinals κ.*
2. *If T has* IP, *then $f_T(\kappa) = 2^\kappa$ for all infinite cardinals κ.*

- *Assuming GCH,* $\operatorname{ded}\kappa = 2^\kappa$ *for all κ. On the other hand:*
- *[Mitchell] For every cardinal κ with* $\operatorname{cf}(\kappa) > \aleph_0$, *there is a forcing extension of the model of ZFC such that* $(\operatorname{ded}\kappa)^{\aleph_0} < 2^\kappa$.

A proof for NIP, 2

- The actual result in the original paper of Mekler is:

Fact

$f_{\mathrm{Th}(G(\mathcal{M}))}(\kappa) \leq f_{\mathrm{Th}(\mathcal{M})}(\kappa) + \aleph_0$ *for all infinite cardinals* κ.

- Hence if $\mathrm{Th}(\mathcal{M})$ is NIP, then $f_{\mathrm{Th}(G(\mathcal{M}))}(\kappa) \leq (\mathrm{ded}\,\kappa)^{\aleph_0}$ for all κ, in all models of ZFC.
- Combining with Mitchell and using Schoenfield's absoluteness, $\mathrm{Th}(G(\mathcal{M}))$ is NIP.

- Admittedly this is somewhat esoteric, and more importantly doesn't generalize to $k > 1$.

Characterization of k-dependence

- We want a formula-free characterization of k-dependence (in $\mathrm{Th}(G(\mathcal{M}))$ we understand automorphisms, but not formulas).
- Let $\kappa := |T|^+$.

Fact

T is NIP $\iff$ for every ($\emptyset$-)indiscernible sequence $(a_i : i \in \kappa)$ and b of finite tuples in $\mathbb{M}$, there is some $\alpha \in \kappa$ such that $(a_i : i > \alpha)$ is indiscernible over b.

- What is the analogue for k-dependence?

Generalized indiscernibles

- T is a theory in a language $\mathcal{L}$, $\mathbb{M} \models T$.

Definition

Let I be an $\mathcal{L}_0$-structure. Say that $\bar{a} = (a_i : i \in I)$, with a_i a tuple in $\mathbb{M}$, is *I-indiscernible over $C \subseteq \mathbb{M}$* if for all $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ from I:

$$\mathrm{qftp}_{\mathcal{L}_0}(i_1, \ldots, i_n) = \mathrm{qftp}_{\mathcal{L}_0}(j_1, \ldots, j_n) \implies$$

$$\mathrm{tp}_{\mathcal{L}}(a_{i_1}, \ldots, a_{i_n}/C) = \mathrm{tp}_{\mathcal{L}}(a_{j_1}, \ldots, a_{j_n}/C).$$

- For $\mathcal{L}_0$-structures I, J, say that $(b_j : j \in J)$ is *based on* $(a_i : i \in I)$ *over* C if for any finite set Δ of $\mathcal{L}(C)$-formulas and any $(j_0, \ldots, j_n)$ from J there is some $(i_1, \ldots, i_n)$ from I s.t. $\mathrm{qftp}_{\mathcal{L}_0}(j_1, \ldots, j_n) = \mathrm{qftp}_{\mathcal{L}_0}(i_1, \ldots, i_n)$ and $\mathrm{tp}_{\Delta}(b_{j_1}, \ldots, b_{j_n}) = \mathrm{tp}_{\Delta}(a_{i_1}, \ldots, a_{i_n})$.
- We say that *I-indiscernibles exist* if for any $\bar{a}$ indexed by I there is an I-indiscernible based on it.

Connection to structural Ramsey theory

- Implicitly used by Shelah already in the classification book, made explicit by Scow and others.

Definition

Let K be a class of finite $\mathcal{L}_0$-structures. For $A, B \in K$, let $\binom{B}{A}$ be the set of all $A' \subseteq B$ s.t. $A' \cong A$.
K is *Ramsey* if for any $A, B \in K$ and $k \in \omega$ there is some $C \in K$ s.t. for any coloring $f : \binom{C}{A} \to k$, there is some $B' \in \binom{C}{B}$ s.t. $f \restriction \binom{B'}{A}$ is constant.

- Classical Ramsey theorem $\iff$ the class of finite linear orders is Ramsey.

Fact

Let K be a Fraïssé class, and let I be its limit. If K is Ramsey, then I-indiscernibles exist.

Ordered random hypergraph indiscernibles

Fact

[Nesétril, Rödl '77,'83] For any $k \in \omega$, the class of all finite ordered k-hypergraphs is Ramsey.

- Fix $k \in \omega$. Modifying their proof, we have existence of $\mathcal{G}$-indiscernibles for $\mathcal{G} = (P_1, \ldots, P_k, R(x_1, \ldots, x_k), <)$ the ordered k-partite random hypergraph (where $P_1 < \ldots < P_k$).
- Let $\mathcal{O} = (P_1, \ldots, P_k, <)$ denote the reduct of $\mathcal{G}$.
- Of course, $(a_g : g \in \mathcal{G})$ is $\mathcal{O}$-indiscernible $/C$ implies it is $\mathcal{G}$-indiscernible $/C$.
- Clarifying Shelah,

Fact

[C., Palacin, Takeuchi '14] TFAE:

1. *T is k-dependent.*
2. *For any $(a_g : g \in \mathcal{G})$ and b, with a_g, b finite tuples in $\mathbb{M}$, if $(a_g : g \in \mathcal{G})$ is $\mathcal{G}$-indiscernible over b and $\mathcal{O}$-indiscernible (over $\emptyset$), then it is $\mathcal{O}$-indiscernible over b.*

Mekler's construction in more detail, 1

- A graph (binary, symmetric, irreflexive relation) C is *nice* if:
  - $\exists a \neq b$,
  - $\forall a \neq b \exists c \left(R\left(a,c\right) \wedge \neg R\left(b,c\right)\right)$,
  - no triangles or squares.

Fact

Any structure in a finite relational language is bi-interpretable with a nice graph.

- Let $G \models \mathrm{Th}\left(G\left(C\right)\right)$, where $G\left(C\right)$ is generated freely by the vertices of C, and two generators commute $\iff$ they are connected by an edge in Cs.
- We consider the following $\emptyset$-definable equivalence relations on G, each refining the previous one:
  - $g \sim h \iff C_G\left(g\right) = C_G\left(h\right)$,
  - $g \approx h \iff \exists r \in \omega, c \in Z\left(G\right)$ s.t. $g = h^r c$.
  - $g \equiv_Z h \iff gZ\left(G\right) = hZ\left(G\right)$.

Mekler's construction in more detail, 2

- $g \in G$ is *of type q* if $\exists$ q-many $\approx$-classes in $[g]_\sim$.
- g is *isolated* if $[g]_\approx = [g]_{\equiv_Z}$.
- G can be partitioned into the following $\emptyset$-definable set:
  - non-isolated elements of type 1 — type 1^ν,
  - isolated elements of type 1 — type 1^ι,
  - elements of type p,
  - elements of type $p-1$.
- For every $g \in G$ of type p, the elements of G commuting with it are:
  - elements $\sim$-equaivalent to g,
  - an element b of type 1^ν together with the elements $\sim$-equivalent to b.
- Such a b is called a *handle of g*, and is definable from g up to $\sim$-equivalence.

Mekler's construction in more detail, 3

Definition

A set $X \subseteq G$ is a *transversal* if $X = X_\nu \sqcup X_p \sqcup X_\iota$, where:

1. X_ν: representatives for each $\sim$-class of elements of type 1^ν in G;
2. X_p: representatives of $\sim$-classes of *proper* (i.e. not a product of any elements of type 1^ν) elements of type p, maximal with the property that if $Y \subseteq X_p$ is a finite of elements with the same handle, then Y is independent modulo the subgroup generated by all elements of type 1^ν and $Z(G)$;
3. X_ι: representatives of $\sim$-classes of proper elements of type 1^ι, maximal independent modulo the subgroup generated by all elements of types 1^ν and p in G, together with $Z(G)$.

Mekler's construction in more detail, 4

- $C = (V, R)$ is interpreted in G as $\Gamma(G)$:
  - $V = \{g \in G : g \text{ is of type } 1^\nu, g \notin Z(G)\} / \approx$,
  - $\left([g]_\approx, [h]_\approx\right) \in R \iff g, h$ commute.
- For X a transversal of G, $\Gamma(X_\nu)$ is isomorphic to C.
- Let $G \models \mathrm{Th}(G(C))$ and X a transversal of G. There is a subgroup (elementary abelian p-group) H of $Z(G)$ s.t. $G \cong \langle X \rangle \times H$.
- There is some canonicity about this choice: $\langle X \rangle' = G'$ for any transversal X of G.

Mekler's construction in more detail, summarizing

- For any partial transversal X' and any linearly independent over G' subset H' of $Z(G)$, we can find a transversal $X \supseteq X'$ and a maximal set $H \supseteq H'$ s.t. $G = \langle X \rangle \times \langle H \rangle$.
- **Lemma.** Both conditions on X' and H' are type-definable.
- If $Y, Z \subseteq X$ and $h : Y \to Z$ is a bijection respecting the 1^ν-, p-, and 1^ι-parts and the handles, and $\mathrm{tp}_\Gamma(Y_\nu) = \mathrm{tp}_\Gamma(h(Y_\nu))$, then $\mathrm{tp}_G(Y) = \mathrm{tp}_G(h(Y))$.
- Moreover, assuming saturation, h extends to an automorphism of G by gluing it with any automorphism of $\langle H \rangle$.

Sketch of the proof, 1

- Let $G \models \mathrm{Th}(G(\mathcal{M}))$ be a monster model, and $\phi(x; y_1, \ldots, y_k)$ not k-dependent.
- Choose a transversal X and $H \subseteq Z(G)$ s.t. $G = \langle X \rangle \times \langle H \rangle$.
- Compactness: a very large witness $(a_g : g \in \mathcal{G})$ to the failure of k-dependence, shattered by ϕ.
- For cardinality reasons, may assume $a_g = t(\bar{x}_g, \bar{h}_g)$ for some $\mathcal{L}_G$-term t and $\bar{x}_g$ from X and $\bar{h}_g$ from H.
- Can close under handles and, changing the formula, replace the original shattered set by $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$.
- Using type-definability of partial transversals, etc. and existence of $\mathcal{G}$-indiscernibles, can assume $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is $\mathcal{O}$-indiscernible (possibly changing the transversal to some X', H').
- As $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is shattered, can choose $b = s(\bar{y}, \bar{k}) \in G$ with $\bar{y} \in X', \bar{k} \in H'$ s.t. $\phi(b; y_1, \ldots, y_k)$ cuts out *exactly* the edge relation of the random k-hypergraph $\mathcal{G}$.

Sketch of the proof, 2

- Using existence of $\mathcal{G}$-indiscernibles again, can assume that $\left(\bar{x}_g \bar{h}_g : g \in \mathcal{G}\right)$ is $\mathcal{G}$-indiscernible over b (needs some argument, replacing X', H' by some X'', H'').
- Using that $\mathrm{Th}\left(\langle X \rangle\right)$ and $\mathrm{Th}\left(\langle H \rangle\right)$ are k-dependent by assumption (hence $\mathcal{G}$-indiscernibility collapses to $\mathcal{O}$-indiscernibility in them by the characterization above), can build an automorphism of G (glueing separate automorphisms of $\langle X'' \rangle$ and $\langle H'' \rangle$ together by the lemma above) σ such that:
- for some finite tuples of indices $\bar{g}, \bar{h}$ of the same type in $\mathcal{O}$, but **not** in $\mathcal{G}$, σ fixes b and sends $\left(\bar{x}_g \bar{h}_g : g \in \bar{g}\right)$ to $\left(\bar{x}_h \bar{h}_h : h \in \bar{h}\right)$.
- — contradiction to the choice of b.

Other results and directions

Theorem

[C., Hempel '17] $\mathrm{Th}(\mathcal{M})$ *is* NTP_2 $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ *is* NTP_2.

- **Problem.**
  - Are there pseudofinite strictly k-dependent groups?
  - Are there ω-categorical strictly k-dependent groups?