Fractional Helly property in model theory

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Helly’s theorem

Theorem

[Helly, 1913] Let $S_1, \ldots, S_n$ be convex subsets of $\mathbb{R}^d$, with $n > d$. If the intersection of every $d + 1$ of these sets is non-empty, then the intersection of the whole collection $\bigcap_{i=1}^{n} S_i$ is non-empty.
Theorem

[Katchalski, Liu, 1979] Fix dimension \( d \geq 1 \). The for every \( \alpha \in (0, 1) \) there exists \( \beta = \beta(\alpha, d) \in (0, 1) \) such that the following holds:

If \( S_1, \ldots, S_n \) are convex sets in \( \mathbb{R}^d \), \( n \geq d + 1 \), such that \( \bigcap_{i \in I} S_i \neq \emptyset \) for at least \( \alpha \binom{n}{d+1} \) of the sets \( I \in \binom{[n]}{d+1} \), then there is some \( J \subseteq [n] \) such that \( |J| \geq \beta n \) and \( \bigcap_{i \in J} S_i \neq \emptyset \).
Let now \((X, \mathcal{F})\) be an arbitrary set system (i.e. \(X\) is a set and \(\mathcal{F}\) is a family of subsets of \(X\)).

**Definition**

We say that \(\mathcal{F}\) satisfies the *fractional Helly property*, or FHP, if there is some \(d \in \mathbb{N}\) such that for every \(\alpha \in (0, 1]\) there exists \(\beta \in (0, 1]\) satisfying the following:

If \((S_1, \ldots, S_n) \in \mathcal{F}^n\) is such that \(\bigcap_{i \in I} S_i \neq \emptyset\) for at least \(\alpha \binom{n}{d}\) of the sets \(I \in \binom{[n]}{d}\), then there is some \(J \subseteq [n]\) such that \(|J| \geq \beta n\) and \(\bigcap_{i \in J} S_i \neq \emptyset\).

The minimal \(d\) for which this holds is the *fractional Helly number* for \(\mathcal{F}\) (if this holds for \(d\), then also holds for any \(d' \geq d\)).
FHP for formulas

Let $T$ be a complete first-order theory in a language $\mathcal{L}$, $\mathcal{M} \models T$ and $\phi(x, y) \in \mathcal{L}$ a formula.

We associate with it a definable family $\mathcal{F}_\phi = \{\phi(M, b) : b \in M_y\}$ of subsets of $M_x$.

Definition
We say that $\phi(x, y)$ has FHP if the family $\mathcal{F}_\phi$ has FHP (and the fractional Helly number of $\phi$ is the fractional Helly number of $\mathcal{F}_\phi$). $T$ has FHP if every formula has FHP.
(Note: FHP is a property of the theory, rather than of the specific model $\mathcal{M}$).
FHP and Shelah’s classification

- FHP implies $\text{NTP}_2$.
- FHP implies low.
- [Matousek, 2003] NIP implies FHP. More precisely, if $
abla^*_{\mathcal{F}}(n) = o(n^d)$ as $n \to \infty$ (e.g. if $\text{vc}^*(\mathcal{F}) < d$), then $d$ is a fractional Helly number for $\mathcal{F}$.
- If all formulas $\phi(x, y)$ with $|x| = 1$ have FHP, then $T$ has FHP.
- There is no implication between FHP and wnfcp.
FHP relatively to a class of measures

- Recall: a Keisler measure $\mu$ on $M_y$ is a finitely additive probability measure on the Boolean algebra of definable subsets of $M_y$.

- Let $\mathcal{M}$ be a class of measures on $M_y$ such that for every $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in \mathcal{M}$ we fix a certain product measure $\mu^*$ on $M_{n \times y}$. If $\mu_i = \mu$ for $i = 1, \ldots, n$ we denote $\mu^*$ by $\mu^{(n)}$.

**Definition**

We say that $\phi(x, y)$ has FHP relatively to $\mathcal{M}$ if there is some $d \in \mathbb{N}$ such that for any $\alpha > 0$ there is $\beta > 0$ satisfying: for any $\mu \in \mathcal{M}$, if $\mu^{(d)} \left( \exists x \bigwedge_{i=1}^d \phi(x, y_i) \right) \geq \alpha$ then there is some $a \in M_x$ with $\mu(\phi(a, y)) \geq \beta$.

- Note: $\phi$ has FHP iff it has FHP relatively to the class $\mathcal{M}_{\text{fin}}$ of finitely supported measures (with the unique product measure).
Fap measures, 1

Definition

Let $\mu$ be a measure on $M_x$ and $\phi(x, y) \in \mathcal{L}$.

1. Given $\varepsilon > 0$, a multiset $A = \{a_1, \ldots, a_n\} \subseteq M_x$ is an $\varepsilon$-approximation of $\mu$ on $\phi$ if for every $b \in M_y$,
   
   \[
   \mu(\phi(x, b)) \approx \varepsilon \frac{|\{i : \models \phi(a_i, b)\}|}{n}.
   \]

2. $\mu$ is finitely approximated on $\phi$, or fap on $\phi$, if it admits a finite $\varepsilon$-approximation on $\phi$ for all $\varepsilon > 0$.

3. $\mu$ is fap if it is fap on every $\phi$.

▶ In NIP, $\mu$ is fap iff $\mu$ is generically stable.

▶ Examples of fap measures:
  
  ▶ A measure concentrated on a finite set,
  
  ▶ In an o-minimal $M$, Lebesgue measure on $[0, 1]$ (restricted to definable sets),
  
  ▶ In $\mathbb{Q}_p$, additive Haar measure on the compact ball $\mathbb{Z}_p$.

▶ The $\{0, 1\}$-measure given by the type at $+\infty$ in $(\mathbb{R}, +, \times, <, 0, 1)$ is not fap.
Fap measures, 2

- Assume we are given a definable relation \( E(x, y) \in \text{Def}(M_{xy}) \).
- Let \( \mu \) and \( \nu \) be Keisler measures on \( M_x \) and \( M_y \), respectively.
- Note that \( \text{Def}(M_{xy}) \neq \text{Def}(M_x) \times \text{Def}(M_y) \), and \( E \) may not be \( \mu \times \nu \)-measurable.
- In general, there are many ways to extend the product measure \( \mu \times \nu \) to a measure \( \omega \) on \( \text{Def}(M_{xy}) \).
- For fap measures, we have a canonical choice.

**Definition.** Given fap measures \( \mu, \nu \), on \( M_x, M_y \) respectively, we define a measure \( \mu \otimes \nu \) on \( M_{xy} \) by

\[
\mu \otimes \nu (E(x, y)) = \int_{M_x} \left( \int_{M_y} 1_E(x, y) \, d\nu \right) \, d\mu.
\]

- It is well-defined, fap and satisfies the Fubini property:

\[
\mu \otimes \nu = \nu \otimes \mu.
\]
FHP relatively to fap measures

Lemma
If \( \phi \) has FHP (i.e., relatively to the class \( \mathcal{M}_{\text{fin}} \)), then \( \phi \) has FHP relatively to the class \( \mathcal{M}_{\text{fap}} \) of fap measures (with \( \otimes \)).

- Then Matousek’s theorem + Fact immediately imply (taking contrapositives):

Fact
[Hrushovski, Pillay, Simon, “A note on generically stable measures and fsg groups”] Let \( T \) be NIP and \( \mu \) a generically stable measure. If \( \mu(\phi(x, b)) = 0 \) for all \( b \), then there is some \( d \) such that
\[
\mu^{(d)}(\exists y (\phi(x_1, y) \land \ldots \land \phi(x_d, y))) = 0.
\]
- Conversely, under the global NIP assumption Matousek’s theorem follows from the fact using that the class of generically stable measures in NIP is closed under ultraproducts.
No reason to fix only one measure. We have (refining [Pillay, “Weight and measure in NIP theories”]):

**Theorem**

Let $T$ be NIP, such that $dp\text{-}\text{rank}("x = x") \leq d$. Then for any formulas $\phi_1(x, y_1), \ldots, \phi_{d+1}(x, y_{d+1}) \in \mathcal{L}$ and $\alpha > 0$ there is some $\beta > 0$ such that:

if $\mu_i$ is a fap measure on $M_{y_i}$, $i = 1, \ldots, d + 1$, and

$\mu_1 \otimes \ldots \otimes \mu_{d+1} \left( \exists x \bigwedge_{i=1}^{d+1} \phi_i(x, y_i) \right) \geq \alpha$ then there is some $a \in M_x$ and $1 \leq i \leq d + 1$ such that $\mu_i(\phi_i(a, y_i)) \geq \beta$.

This corresponds to the so called “colorful fractional Helly property” for NIP families (was known in combinatorics for convex sets in $\mathbb{R}^k$, due to [Barany et. al., 2014]).

**Corollary**

The fractional Helly number of $\phi(x, y)$ is at most $dp\text{-}\text{rank}(x = x) + 1$ (by the Theorem for $\phi_i(x, y_i) = \phi(x, y)$ and $\mu_i = \mu$).
Definition
We say that a family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ is pierceable if there is some number $d \in \omega$ such that for any $q \geq p \geq d$ there is some $N = N(p, q) \in \omega$ such that if a finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ satisfies the $(p, q)$-property (i.e. among any $q$ sets from $\mathcal{F}$, at least $p$ have a non-empty intersection), then there are some $a_1, \ldots, a_N \in X$ such that every $S \in \mathcal{F}'$ contains at least one of the $a_i$'s.

Theorem
[Alon, Kleitman] For any $d$, $\mathcal{F} = \{\text{convex subsets of } \mathbb{R}^d\}$ is pierceable.

Theorem
[Matousek] If $\text{VC}(\mathcal{F}) < \infty$, then $\mathcal{F}$ is pierceable.

- The proof combines FHP + existence of $\varepsilon$-nets.
Theorem
Assume that $VC(\mathcal{F}) = \infty$. Then the family $\mathcal{F}' = \{S_1 \land \neg S_2 : S_1, S_2 \in \mathcal{F}\}$ is not pierceable.

▶ In particular, $T$ is NIP iff $\mathcal{F}_\phi$ is pierceable for every $\phi \in \mathcal{L}$.
▶ The family of convex sets in $\mathbb{R}^d$ shows that this doesn’t hold at the level of a formula.
FHP in MS-measurable structures, 1

Definition
[Macpherson, Steinhorn, 2008] An $L$-structure $M$ is \textit{MS-measurable} if for every non-empty set $X \subseteq M^n$ definable with parameters, we have a pair $(\dim(X), \text{meas}(X))$ with $\dim(X) \in \mathbb{N}$, $\dim(X) \leq n$ and $\text{meas}(X) \in \mathbb{R}_{>0}$ satisfying some strong definability properties and a \textbf{Fubini} condition.

Example
[Chatzidakis, van den Dries, Macintyre] Let $M = \prod_{p \in P} \mathbb{F}_p / \mathcal{U}$ be an ultraproduct of finite fields, $P$ a set of primes, $\mathcal{U}$ a non-principal ultrafilter on $P$.
Let $X \subseteq M^n$ be a definable set, $X = \prod_{p \in P} X_p / \mathcal{U}$ for some $X_p \subseteq \mathbb{F}_p^n$. Then $(\dim(X), \text{meas}(X)) = (d, \alpha)$ if

$$|X_p| \approx \alpha p^d$$

for $\mathcal{U}$-many $p$. 
FHP in MS-measurable structures, 2

For any definable set $B \subseteq M_y$, we have a Keisler measure $\mu_B$ concentrated on $B$ and defined by

$$\mu_B (X) = \begin{cases} \frac{\text{meas}(X \cap B)}{\text{meas}(B)} & \text{if } \dim (X \cap B) = \dim (B) , \\ 0 & \text{if } \dim (X \cap B) < \dim (B) \end{cases}$$

for all definable $X \subseteq M_y$.

Let $M = \{\mu_B : B \subseteq M_y \text{ definable}\}$, and we take $\mu_{B_1} \otimes \mu_{B_2} := \mu_{B_1 \times B_2}$ (given by the same double integral as in fap measures).

Theorem

In an MS-measurable theory, $\phi (x, y)$ satisfies FHP relatively to the class of measures $M_y = \{\mu_B : B \subseteq M_y \text{ definable}\}$.

In particular, MS-measurable implies FHP (as $\mu_B$ with $B$ finite is the counting measure concentrated on $B$), and the fractional Helly number of $\phi (x, y)$ is at most $\max \{\dim \phi (x, b) : b \in M_y\} + 1$.

Gives a definable FHP theorem for large finite fields.
Ultraproducts of the $p$-adics

- For each prime $p$, the field $\mathbb{Q}_p$ is NIP, so satisfies FHP relatively to generically stable measures.

**Theorem**

Let $\mathcal{M} = \prod_{p \in P} \mathbb{Q}_p/\mathcal{U}$ for $P$ a set of primes, $\mathcal{U}$ a non-principal ultrafilter on $P$. Then $T = \text{Th}(\mathcal{M})$ satisfies FHP (i.e. for finitely supported measures).

- Problem: is there a motivic version?

- Note: $T$ is neither NIP nor simple. Previously known: $T$ is NTP$_2$ (C.) and moreover $T$ is inp-minimal (C., Simon).
Some more examples

Theorem

1. Th(\(\mathbb{Z}, +, \text{Sqf}\)) is FHP (elaborating on the results of Bhardwaj-Tran).

2. Assuming Dickson’s conjecture, Th(\(\mathbb{Z}, +, \text{Pr}\)) is not FHP (but it is supersimple of SU-rank 1 by Kaplan-Shelah, hence wnfcp).
Let $G$ be a group and $A \subseteq G$.

$A$ is generic \implies A$ is weakly generic $G$-amenable $\implies \mu(A) > 0$ for some $G$-invariant measure $\mu$ $\implies A$ is $f$-generic.

[C., Simon] In definably amenable NIP, $f$-generic = weak generic.

**Theorem**

*Let $G$ be an amenable group with FHP and $A \subseteq G$ definable. TFAE:*

1. $A$ is $f$-generic.
2. $\mu(A) > 0$ for some $G$-invariant Keisler measure.
3. $\mu(A) > 0$ for some $G$-invariant measure on $\mathcal{P}(G)$.

Problem: does $(1) \iff (2)$ hold assuming only that $G$ is definably amenable?
Example

1. In $\text{Th}(\mathbb{Z}, +, \text{Pr})$, $\text{Pr}$ is $f$-generic, but $\mu(\text{Pr}) = 0$ for any invariant Keisler measure (by the Prime Number theorem).

2. In $\text{Th}(\mathbb{Z}, +, \text{Sqf})$, $\mu(\text{Sqf}) = \frac{6}{\pi^2} > 0$ for an invariant measure (Banach density), but $\text{Sqf}$ is not weakly generic.