Distality in Valued Fields and Related Structures

joint w/ Aschenbrenner, Gehret and Ziegler

Main Theorem

Let $K$ be a complete valued field, viewed as $(K, O)$, with value group $\Gamma$ and residue field $k$. Then $K$ is distal (resp., has a distal exp) if $F$

1) $K$ is char $(0, 0)$ or $k$ is char. 0, $k$ is finite, and $O$ is finitely ramified (i.e., $\exists n \in \mathbb{N}$ s.t. $v(n)$ is finite for all $n \geq 1$).

2) both $K$ and $\Gamma$ are distal (resp., have distal expansions).

- "Shefalib's conjecture".
- [Anscombe, Jahnke]
Distality

Fact T completes \( IM = T \). TFAE:

1) For any indisc. seq. \( I = (a_i : i \in \mathbb{Q}) \) and \( b \), if \( (a_i : i \in \mathbb{Q}) \) has indisc./b, then \( I \) is indisc./b.

Thus\

2) "Strong honest def": For every \( p(x,y) \), there exists \( \psi(x_1,y_1,...,y_n) \) such that\:

\[
\forall B \subseteq |M|, a \in |M| \}, \exists b_1,...,b_n \in B \text{ s.t.} \psi(a,b_1,...,b_n) = \psi(a,b) + t p(a/b),
\]

3) Definable strong Erdős–Hajnal property:

\[\forall \varphi(x,y) \exists \varepsilon > 0, \varphi_1(x,z), \varphi_2(y,z) \text{ s.t.} \]

\[
\forall A \subseteq |M|, B \subseteq |M| \}, \exists z_i \in |M| \text{ s.t.} \]

\[
|\varphi_1(A,z_i)| \geq \varepsilon |A|, |\varphi_2(B,c_2)| \geq \varepsilon |B|, \text{ and}
\]

\[
|\varphi(A,B,c) - \varphi(A,B,c_2)| \leq \varphi(A,B) \text{ or } c \leq A \times B \setminus \varphi(A,B).
\]

1) is def. of distality \([\text{Simon}]\)

1) \( \Rightarrow 2) \) \([\text{C., Simon}])

\( \Rightarrow 3) \) \([\text{C., Starchenko}]. 1) \Rightarrow \text{NIP}

Ex: o-min., p-adics, ... In 1), wlog \( a_i \) or \( b \) is a singleton.
Prop \[\text{Let } T \text{ be NIP, } D \text{ is } \varphi \text{-definable with } D \text{ ind is distal}\]

\[\left(\langle D, R, \ldots \rangle, R = D^n \cap E, E \varphi \text{-def} \right)\]

1. For any \( b \in M \) and \( I = (a_i : i \in \Omega) \in D \), if \( (a_i : i \in \Omega) \) is indisc \( b \), then \( I \) is indisc \( b \).

2. For any \( I = (a_i : i \in \Omega) \in (M \text{ and } b \in D) \), if \( (a_i : i \in \Omega) \) is indisc \( b \), then \( I \) is indisc \( b \).

Cor 1. If \( M \) is NIP, \( D \) is \( \varphi \text{-def} \), with \( D \text{ ind is distal} \)

\[M \leq \text{at} (D) \] then \( M \) is distal.

Cor 2. \( T \) is distal \( \iff T \text{ is distal} \).
Distal fields and rings

Fact \cite{starchenko93} If \( K \) is an \( \text{qf} \)-field def. in a distal \( T \), then \( \text{char}(K) = 0 \).

Reason: the \( qf \)-dep. point-line incidence rel. in \( \text{IF}_{\text{alg}} \) doesn't satisfy \( S^E \).

\[ \begin{align*}
\text{Kaplan, Scanlon, Wagner} \quad & K \text{ is } \text{NIP}, \quad \text{char } p \Rightarrow \text{IF}_{\text{alg}} \leq K, \\
\text{Prob.} \quad & K \text{ is type-def. in a distal str. } \Rightarrow K \text{ char } 0? \\
\text{Prop.} \quad & \text{Supp. } R \text{ is a unital ring w/o zero-divisors def. in a distal } T. \text{ Then } \text{char}(R) = 0 \quad (\text{the smallest } n \geq 1 \text{ s.t. } n.1 = 0, \text{ or } 0 \circ v) \text{.}
\end{align*} \]

\[ \begin{align*}
\text{Prop.} \quad & \text{"no zero-div" can't be dropped.} \\
\text{Prop.} \quad & \text{H=IF}_{\text{alg}} \text{ has a distal exp (namng val)} \\
& R := \text{IF}_{\text{alg}} \times \text{H - comm. ring } \text{ of char } p. \\
\text{Work in progress w/ Simon: all ab. gps have dist exps.} \\
\text{Fact:} \quad & \text{If } (K,0) \text{ is NIP and } k \text{ is finite, then } v \text{ is finitely ramified.}
\end{align*} \]
So: If \((K, O)\) has a dist. exp., then \((K, O)\) is fin. ram. and \(k\) has char 0 or is finite.

Reduction to \(RV\) \(r\) \(\delta\) \(\Lambda\) \(\Gamma\) \(\Sigma\) \(\Theta\)

For \(\delta \in \Gamma \geq 0\), let \(m_\delta = \{x \in k : v(x) \geq \delta\}\), an ideal of \(O\).

\(RV_\delta := k/(1+m_\delta)\), \(rv_\delta : k \to RV_\delta\)

\(\nu_{rv} RV = RV_0\) \(\nu_{rv} \rightarrow \Gamma \rightarrow 0\)

\(SES\) on ab. grps: \(1 \to k^\times \to RV^\times \xrightarrow{\nu_{rv}} \Gamma \to 0\)

on \(RV_\delta\) \(\Theta_\delta (r, s, t) \iff \exists x, y, z \in k \ (r = rv_\delta(x) \land s = rv_\delta(y) \land t = rv_\delta(z) \land x + y = z)\)

\(\overline{RV} = \begin{cases} (RV_0, \Theta_0, \times) & \text{if char } k = 0 \\ (RV_\delta, \Theta_\delta, \times) & \text{if char } k = \rho. \end{cases}\)

Note: \(RV\) is interp. in \((K, O)\).

Fact [Flenner]
1) char \((k) = 0\), hence \(\Rightarrow (k, O)\) has QE down to \(RV\)

2) \(RV\) is fully stably embedded.

Rem: holds "responderently!"
Using analysis of indisc. seq.

Prop

\( (K, 0) \) is distal (has dist exp) \( \iff \) it is fin. ram.

and \( \overline{RV} \) is dist (has dist exp).

Reduction to \( k \) and \( \mathbb{F} \)

Prop

\( \overline{RV} \) is distal \( \iff \) \( k \) and \( \mathbb{F} \) are \( k \)-finite.

char \( k > 0 \) - treated by the finite covers preserve dist.

Lemma

\[ RV \rightarrow RV_v(p^2) \rightarrow RV_v(p) \]

char \( 0 \) - \( RV \) is a reduct of \( k \times \mathbb{F} \).
Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a pure SES of abelian groups.

- $i(A)$ is a pure subgroup of $B$.
- If $ax = a$ has a sol. in $B$, then has a sol. in $i(A)$.

Add sorts:

1. $\forall n A / n A$

2. $\forall n : A \to A / n A$

3. $\forall n : B \to A / n A$ on $\pi^{-1}(nC)$.

$\pi^{-1}(nC) = nB + i(A) \Rightarrow (nB + i(A))/nB \xrightarrow{\sim} i(A)/(nBn^{-1}i(A))$

$\Rightarrow A / n A$

outside of $\pi^{-1}(nC)$ - $0$.

$\forall n : \pi_n = \pi \circ \pi^{-1}$ on $\pi^{-1}(nC)$.

1. If SES splits:

2. If $\pi_n = \pi \circ \pi^{-1}$ on $\pi^{-1}(nC)$.

LAC - the group lang on $A$ and on $C$, with arb. add struct.

LB - the grp. lang on $B$

Thm: modulo the theory of pure SES, every $\gamma'(X_A, X_B, X_C)$ is equiv. $\gamma'(X_A, \bar{\pi}(X_B), \ldots)$.

$\delta m(X_B, X_C)$ where $\delta m$ is given by $\delta m \cdot (X_B, X_C) \Rightarrow \gamma(\pi(X_A), \bar{\pi}(X_B), \ldots)$ and $\delta m(X_B) = \{ \delta m(t(X_B)) \}$ are $h_B$-terms.
Distinguished in OAG's

Thus let \((G^{1:2})\) be a str. dep. OAG,

\[ G \text{ is distal } \iff G \text{ is dp-min} \iff \|G/pG\| < \infty \quad \forall \text{ prime } p. \]

Conj: All OAG's have distal exp's.

"Shelah's conj" + \(\sqrt{\phantom{a}}\) \(\rightarrow\) An NIP field \(K\) is distal

If:

- \(K\) is a hens. val.
- with res. field \(ACF_0\), \(RCF\) or
  - finite
  - \(\phi\)
- \(K\) doesn't interp. an infr. field of positive char.

Ramanujan graph. [Nesetril, Ossona de Mendez; ...]

[Bexel, Kestner] \(M\) is distal \(\iff\) \(M^{\Sigma^n}\) is distal.