Invariant measures in simple theories

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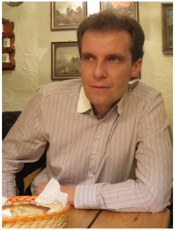
- Joint work with Ehud Hrushovski, Alex Kruckman, Krzysztof Krupinski, Slavko Moconja, Anand Pillay and Nick Ramsey






Spaces of types

- Let T be a complete first-order theory in a language $\mathcal{L}$, $\mathbb{M} \models T$ a monster model (i.e. κ-saturated and κ-homogeneous for a sufficiently large cardinal κ), $\mathcal{M} \preceq \mathbb{M}$ a small elementary submodel.
- For $A \subseteq \mathbb{M}$ and x an arbitrary tuple of variables, $S_x(A)$ denotes the set of complete types over A.
- Let $\mathcal{L}_x(A)$ denote the set of all formulas $\varphi(x)$ with parameters in A, up to logical equivalence — which we identify with the Boolean algebra of A-definable subsets of $\mathbb{M}_x$; $\mathcal{L}_x := \mathcal{L}_x(\emptyset)$.
- Then the types in $S_x(A)$ are the ultrafilter on $\mathcal{L}_x(A)$.
- By Stone duality, $S_x(A)$ is a totally disconnected compact Hausdorff topological space with a basis of clopen sets of the form
$$\langle \varphi \rangle := \{p \in S_x(A) : \varphi(x) \in p\}$$
for $\varphi(x) \in \mathcal{L}_x(A)$.
- We refer to types in $S_x(\mathbb{M})$ as *global types*.

Keisler measures

- A *Keisler measure* μ in variables x over $A \subseteq \mathbb{M}$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of A-definable subsets of $\mathbb{M}_x$.
- $\mathfrak{M}_x(A)$ denotes the set of all Keisler measures in x over A.
- Then $\mathfrak{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A), r_i, s_i \in [0,1]$ for $i < n$.
- Identifying p with the Dirac measure δ_p, $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathfrak{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$.

Some examples of Keisler measures, 1

1. In arbitrary T, given $p_i \in S_x(A)$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ with $\sum_{i \in \mathbb{N}} r_i = 1$, $\mu := \sum_{i \in \mathbb{N}} r_i \delta_{p_i} \in \mathfrak{M}_x(A)$.
2. Let $T = \mathrm{Th}(\mathbb{N}, =)$, $|x| = 1$. Then

$$S_x(\mathbb{M}) = \{\mathrm{tp}(a/\mathbb{M}) : a \in \mathbb{M}\} \cup \{p_\infty\},$$

   where p_∞ is the unique non-realized type axiomatized by $\{x \neq a : a \in \mathbb{M}\}$. By QE, every formula is a Boolean combination of $\{x = a : a \in \mathbb{M}\}$, from which it follows that every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is as in (1).
3. More generally, if T is ω-stable (e.g. strongly minimal, say ACF_p for p prime or 0) and x is finite, then every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is a sum of types as in (1) (for T stable — holds locally).
4. Let $T = \mathrm{Th}(\mathbb{R}, <)$, λ be the Lebesgue measure on $\mathbb{R}$ and $|x| = 1$. For $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, define $\mu(\varphi) := \lambda\left(\varphi(\mathbb{M}) \cap [0,1]_{\mathbb{R}}\right)$ (this set is Borel by QE). Then $\mu(X)$ is a Keisler measure, but not a sum of types as in (1).

Some examples of Keisler measures, 2

- If T is NIP, any measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ can be approximated by types, thanks to the VC-theorem (observed in [Pillay, Hrushovski]): for every $\varphi(x, y) \in \mathcal{L}_{x,y}$ and $\varepsilon \in \mathbb{R}_{>0}$ there exist finitely many types $p_1, \ldots, p_n \in S_x(\mathbb{M})$ in the support of μ so that for any $b \in \mathbb{M}_y$,

$$\mu(\varphi(x, b)) \approx^{\varepsilon} \frac{|\{i : \varphi(x, b) \in p_i\}|}{n}.$$

  If μ is *generically stable*, can take all p_i realized types.

- Let $\mathcal{M} = \prod_{i \in \omega} \mathcal{M}_i / \mathcal{U}$ for some finite $\mathcal{M}_i$ and $\mathcal{U}$ a non-principal ultrafilter on ω. For $\varphi(x, a) \in \mathcal{L}_x(\mathcal{M})$ with $a = (a_i : i \in \omega)/\mathcal{U}, a_i \in M_i$, define

$$\mu(\varphi(x, a)) := \lim_{\mathcal{U}} \frac{|\varphi(M_i, a_i)|}{|M_i|}.$$

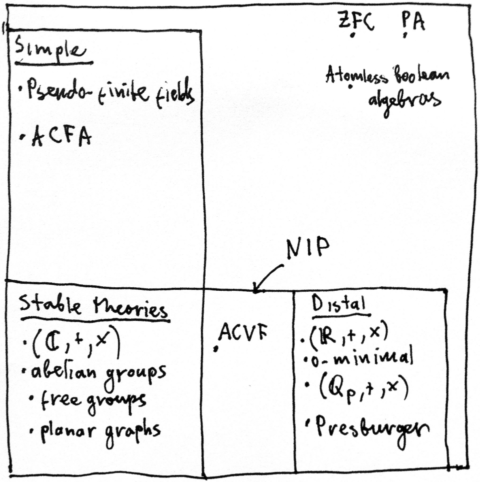
  Then μ is a Keisler measure over $\mathcal{M}$.

Brief history of the theory of Keisler measures

- Measures and forking in stable/NIP theories [Keisler'87]
- Automorphism-invariant measures in ω-categorical structures [Albert'92, Ensley'96]
- Applications to neural networks [Karpinski, Macyntire'00]
- Pillay's conjecture and compact domination [Hrushovski, Peterzil, Pillay'08], [Hrushovski, Pillay'11], [Hrushovski, Pillay, Simon'13]
- Randomizations [Ben Yaacov, Keisler'09] (NIP and stability are preserved)
- Approximate Subgroups [Hrushovski'12]
- Definably amenable NIP groups [C., Simon'15] (in particular translation-invariant measures are classified)
- Tame (equivariant) regularity lemmas: subsets of [C., Conant, Malliaris, Pillay, Shelah, Starchenko, Terry, Tao, Towsner, ...'11– ...]
- See e.g. my review "Model theory, Keisler measures and groups" in BSL, or Starchenko, "NIP, Keisler measures and combinatorics" in Bourbaki.

So measures are the new types in NIP. But outside?

- All of the above — mostly inside NIP theories (thanks to the (equivariant) *VC-theory*, measures are strongly approximated by types). Is there a meaningful theory of measures in *simple theories*?



Forking and dividing

Definition (Shelah)

Suppose A is a set of parameters.

1. $\varphi(x; a)$ *divides over* A if there is an A-indiscernible sequence $(a_i : i < \omega)$ with $a_0 = a$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
2. $\varphi(x; a)$ *forks over* A if

   $$\varphi(x; a) \vdash \bigvee_{i<k} \psi_i(x; c_i),$$

   where each $\psi_i(x; c_i)$ divides over A.
3. A (partial) type p forks or divides over A if it implies a formula that does.
4. Write $a \mathop{\smile\!\!\!\!\!\!|}\,^{f}_{C} b$ to denote that $\mathrm{tp}(a/bC)$ does not fork over C.

Simplicity

Definition (Shelah)

T is simple if $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies local character: for any a and C, there is $B \subseteq C$ with $|B| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\mid}^f_B C$.

Theorem (Kim-Pillay)

The theory T is simple if and only if there is an $Aut(\mathbb{M})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on small subsets of $\mathbb{M}$ satisfying:

1. *Extension, Symmetry, Finite character, Transitivity, Base monotonicity, Local character*
2. *The Independence Theorem (=3-amalgamation): If $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\mid}_M b$, $a' \mathop{\smile\!\!\!\!\!\mid}_M c$ and $b \mathop{\smile\!\!\!\!\!\mid}_M c$, then there is a_* such that $a_* \equiv_{Mb} a$, $a_* \equiv_{Mc} a'$, and $a_* \mathop{\smile\!\!\!\!\!\mid}_M bc$.*

If there is such a relation, it agrees with $\mathop{\smile\!\!\!\!\!\mid}^f$.

Examples of simple theories

1. The random graph.
2. Pseudo-finite fields (more generally, bounded PAC fields) and ACFA.
3. Stable theories with a generic predicate or automorphism.

Measures in simple theories

- Ultraproducts of finite counting measures in pseudofinite fields are very well-behaved, e.g. a strong regularity lemma for definable graphs [Tao] (or more generally, in MS-measurable structures [Hrushovski], [Pillay, Starchenko], [García, Macpherson, Steinhorn]).
- But very few general results outside of NIP so far. Some counterexamples:
  - Independent product $\otimes$ of Borel-definable measures is not associative in general [Conant, Gannon, Hanson'21];
- And some positive results:
  - A generalization of ε-nets for n-dependent theories, and the corresponding regularity lemma approximating relations of any arity by relations of arity n [C.,Towsner] (the case $n = 1$ corresponds to the NIP hypergraph regularity [C., Starchenko]).
  - $NSOP_1$ is preserved under Keisler randomization in continuous logic [Ben Yaacov, C., Ramsey, 21+]

Measures and forking

Definition

Suppose μ is a global Keisler measure. We say μ is *A-invariant* if $\mu(X) = \mu(\sigma(X))$ for all definable sets X (with parameters) and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$. Equivalently, μ is A-invariant if, given any $\varphi(x; y)$ and $b \equiv_A b'$,

$$\mu(\varphi(\mathbb{M}; b)) = \mu(\varphi(\mathbb{M}; b')).$$

Definition

We say a definable set X is *universally of measure zero* over A if $\mu(X) = 0$ for all global A-invariant measures μ. We refer to the collection of sets universally of measure zero as the *universal measure zero ideal*.

Measures and forking

Fact

A formula that forks over A defines a set that is universally of measure zero over A.

Proof.

As a finite union of sets universally of measure zero is universally of measure zero, it suffices to show that if $\varphi(x; a)$ divides over A, then $\mu(\varphi(\mathbb{M}; a)) = 0$. Let $(a_i : i < \omega)$ be an A-indiscernible sequence such that $a_0 = a$ and $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent. If $\mu(\varphi(\mathbb{M}; a)) > 0$ for some A-invariant μ, then there is some maximal k such that $\mu\left(\bigwedge_{i<k} \varphi(x; a_i)\right) > 0$. Then for all $j < \omega$, the sets defined by $\bigwedge_{i<k} \varphi(x; a_{k \cdot j+i})$ have pairwise intersection of measure zero and (by A-indiscernibility and invariance) constant positive measure. This contradicts the fact that μ is a probability measure. $\square$

Definably amenable groups

Definition

Let G be a definable group in some structure (i.e. the set of its elements and the group operation are definable).

- A measure μ on the definable subsets of G is (left) *G-invariant* if $\mu(X) = \mu(g \cdot X)$ for all definable $X \subseteq G$ and $g \in G$.
- G is *definably amenable* if there exists a G-invariant Keisler measure on definable subsets of G.

- Note: there exists a left-invariant measure iff exists a right invariant measure; definable amenability is preserved under elementary equivalence.

Examples of definably amenable groups

- Solvable groups, or more generally any group G such that $G(M)$ is amenable as a discrete group.
- Definable compact groups in o-minimal theories or in p-adics (compact Lie groups, e.g. $SO(3, \mathbb{R})$, seen as definable groups in $\mathbb{R}$).
- Ultraproducts of finite groups.
- Stable groups (in particular the free group F_2, viewed as a structure in a pure group language, is definably amenable).
  Indeed, if G is stable, then G^0 has a unique G^0-invariant generic (global) type p and $G^0 = \mathrm{Stab}(p)$. If X is a definable subset of a coset $g\mathrm{Stab}_\Delta(p)$, for some finite Δ, which is of finite index, then we define

$$\mu(X \cap g\mathrm{Stab}_\Delta(p)) = \begin{cases} \frac{1}{[G:\mathrm{Stab}_\Delta(p)]} & \textit{if } g^{-1}X \in p \\ 0 & \textit{otherwise.} \end{cases}$$

  Then, defining $\mu(X) = \sum_{g \in G/\mathrm{Stab}_\Delta(p)} \mu(X \cap g\mathrm{Stab}_\Delta(p))$, one obtains a G-invariant Keisler measure.

Questions

1. [Harrington] Do the universal measure zero ideal and forking ideals agree in simple theories?
2. [Pillay] Are groups definable in a simple theory always definably amenable?

We develop a general method giving counterexamples for both questions, and discuss here the group case.

Tarski's characterization of amenability

- A *paradoxical decomposition* for a discrete group G consists of pairwise disjoint subsets $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ of G for some $m, n \in \mathbb{N}_{\geq 1}$ and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that G is the union of the $g_i X_i$ and is also the union of the $h_j Y_j$.
- [Tarski] G is amenable if and only if G has no paradoxical decomposition.

An analog for definable amenability, 1

- We fix a definable group G in a structure M.
- By an *(m-)cycle* (for $m \geq 0$) we mean a formal sum $\sum_{i=1,\ldots,m} X_i$ of definable subsets X_i of G. If all the X_i are the same we could write this formal sum as mX_i. We can add such cycles in the obvious way to get the "free abelian monoid" generated by the definable subsets of G. And any definable subset X of G (including G itself) is a (1-)cycle.
- If $X = \sum_{i=1,\ldots,m} X_i$ and $Y = \sum_{j=1,\ldots,n} Y_j$ are two cycles, then by a *definable piecewise* translation f from X to Y we mean a map f from the formal disjoint union $X_1 \sqcup \ldots \sqcup X_m$ to the formal disjoint union $Y_1 \sqcup \ldots \sqcup Y_n$ for which there is a partition of each X_i into definable subsets $X_{i1}, \ldots, X_{in_i}$, and for each i and $t \leq n_i$, an element g_{it} of G such that the restriction $f|X_{it}$ of f to X_{it} is just left translation by g_{it}, and $g_{it}X_{it}$ is a subset of one of the Y_j's.
- A definable piecewise translation f is said to be *injective* if it is injective as a map between formal disjoint unions.

An analog for definable amenability, 2

- We write $X \leq Y$ if there is an injective piecewise definable translation f from X to Y. Note that $\leq$ is reflexive and transitive. Also $X \leq W$ and $Y \leq Z$ implies $X + Y \leq W + Z$.

Definition

By a *definable paradoxical decomposition* of the definable group G we mean an injective definable piecewise translation from $G + Y$ to Y for some cycle Y.

Theorem (Hrushovski, Peterzil, Pillay)

G is definably amenable if and only if G does not have a definable paradoxical decomposition.

- [Corollary] G is not definably amenable iff $(n+1)G \leq nG$ for some $n \geq 1$.
- Tarski's condition corresponds to: $2G \leq G$. It is open if we can always take $n = 2$ in the definable case.

Theorem (C., Hrushovski, Kruckman, Krupinski, Moconja, Pillay and Ramsey'21)

Let T be a model complete theory eliminating $\exists^\infty$ and G a definable group in T. Assume that (in some model) G contains a (not necessarily definable) free group on ≥ 2 generators. Then there exists a model complete expansion T^* of T so that G is not definably amenable in T^* (in fact, $2G \leq G$), and so that if T is simple, then T^* is also simple.

- Example: start with $G := SL_2(\mathbb{C})$ definable in the stable theory ACF_0, obtain a simple (SU-rank 1) theory with a non-definably amenable group.
- The expansion is obtained by adding a "generic" paradoxical decomposition to G. Some interesting tree combinatorics is required to demonstrate that it is axiomatizable, and an explicit description of forking in T^* is obtained in terms of T.

A group example, 1

The language $\mathcal{L}$ will consist of the language of rings, together with 4 quaternary relations C_1, C_2, C_3, C_4. Modulo the theory of integral domains of characteristic zero, we will write SL_2 to denote the definable group of 2×2 matrices of determinant 1. We will treat SL_2 as though it were a sort and C_1, C_2, C_3, C_4 like unary predicates on SL_2. The matrices

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group in $\mathrm{SL}_2(\mathbb{Z})$. Hence so do the matrices

$$a^{-k} b a^k = \begin{pmatrix} 1 - 4k & -8k^2 \\ 2 & 4k + 1 \end{pmatrix},$$

for $k = 0, \dots, 11$. We renumber these 12 matrices in some way as $a(i,j)$ $i \in [4]$, $j \in [3]$. We will refer to the group generated by these matrices as $G := F_{12}$, and we will treat the $a(i,j)$ as though they were individual constants in SL_2 (note that, because they are integer matrices, their entries are given by terms in $\mathcal{L}$).

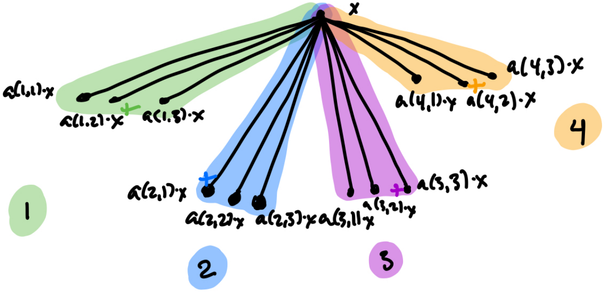
A group example, 2

The universal theory T will extend the theory of integral domains of characteristic zero with a sentence asserting that C_1, C_2, C_3, C_4 form a partition of SL_2, together with the following "coloring" axiom:

$$(\forall x \in \mathrm{SL}_2) \left[\bigwedge_{i \in [4]} \bigvee_{j \in [3]} C_i(a(i,j) \cdot x) \right].$$

We will write a model of T as a pair (R, c) where R is an integral domain of characteristic zero and $c : \mathrm{SL}_2(R) \to [4]$ is a coloring such that $c^{-1}(i) = C_i^R$ for $i \in [4]$.

Coloring axiom



The goal

1. Prove T has a model companion T^* extending the theory of algebraically closed fields of characteristic zero.
2. Prove T^* is simple.
3. Show that SL_2 is a definable group in this theory which is not definable amenable.

Free actions of G

- Suppose $G \curvearrowright X$ is a free action. We may regard X as a disjoint union of Cayley graphs of G.
- For $u, v \in X$, we write $d(u, v)$ for the graph distance from u to v and $B_n(v)$ for the ball of radius n centered at v:

$$B_n(v) = \{u \in X \mid d(v, u) \leq n\}.$$

- Given $V \subseteq X$, we also define

$$B_n(V) = \bigcup_{v \in V} B_n(v).$$

A combinatorial lemma

Definition

Suppose $G \curvearrowright X$ freely and $X_0 \subseteq X$. A coloring $c : X_0 \to [4]$ is *good* if, for all $x \in X_0$ and $i \in [4]$, IF $a(i,j) \cdot x \in X_0$ for all $j \in [3]$, THEN $c(a(i,j) \cdot x) = i$ for some $j \in [3]$.

- The following is a key lemma that will allow us to axiomatize T^*:

Lemma

Suppose G acts freely on X and $X_0 \subseteq X$ is a subset such that $|X_0| = n$ and let $\alpha(n) = 2^{n+1} - 1$. Any good coloring $c : X_0 \to [4]$ that extends to a good coloring $c' : B_{\alpha(n)}(X_0) \to [4]$ also extends to a total good coloring on X.

Safety

Definition

By a *curve*, we mean an absolutely irreducible curve. Suppose $n < \omega$ and $c_0 : [n] \to [4]$ is a function. We say the curve $C \subseteq \mathrm{SL}_2^n$ is *safe for* c_0 if for generic $d = (d_1, \ldots, d_n) \in C$, the coloring $c : \{d_1, \ldots, d_n\} \to [4]$ defined by $c(d_i) = c_0(i)$ extends to a good coloring $c' : B_{\alpha(n)}(\{d_1, \ldots, d_n\}) \to [4]$.

Our combinatorial lemma, and definability of irreducibility in families, gives the following:

Lemma

If $(D_a)_a$ is a definable family of definable subsets in SL_2^n, then for any function $c_0 : [n] \to [4]$ the set $\{a : D_a$ is a curve, safe for $c_0\}$ is definable.

Axiomatizing T^*

The theory T^* is the theory that expresses that an $\mathcal{L}$-structure (K, c) is a model of T^* if K is an algebraically closed field and satisfies the following:

- for every $n < \omega$ and curve $C \subseteq \mathrm{SL}_2^n$, if $c_0 : [n] \to [4]$ is a function, and C is safe for c_0, then there is $d = (d_1, \dots, d_n) \in C(K)$ such that $c_0(i) = c(d_i)$.

As curves in SL_2^n cut out by a fixed number of equations of bounded degree which are safe for a given $c_0 : [n] \to [4]$ form a definable family, by the previous lemma this is indeed axiomatizable, with an axiom for each $n < \omega$ and each such family.

Properties of T^*

- T^* is consistent and is the model completion of T^*.
- If K is a substructure of the monster $(\mathbb{M}, c) \models T$, then tuples a, b satisfy $a \equiv_K b$ (in T^*) if and only if we have $a \equiv_K^{\mathrm{ACF}} b$ and
$$(K(a)^{alg}, c|_{K(a)^{alg}}) \cong_K (K(b)^{alg}, c|_{K(b)^{alg}}).$$
- The theory T^* is simple and $a \mathop{\smile\!\!\!\!\!\!|}_C b$ in T^* if and only if $a \mathop{\smile\!\!\!\!\!\!|}_C^{\mathrm{ACF}} b$. In particular, T has SU-rank 1.

SL_2 is not definably amenable in T^*

Assume that μ is a global Keisler measure on SL_2, invariant under translation. By the coloring axiom, we know that for each $i \in [4]$, we have

$$\mathrm{SL}_2 \subseteq a(i,1)^{-1}C_i \cup a(i,2)^{-1}C_i \cup a(i,3)^{-1}C_i,$$

and, hence, by translation invariance, we have

$$1 \leq 3\mu(C_i),$$

which shows $\mu(C_i) \geq \frac{1}{3}$. On the other hand, because C_1, C_2, C_3, and C_4 partition SL_2, we have

$$1 = \mu(\mathrm{SL}_2) = \sum_{i=1}^{4} \mu(C_i) \geq \frac{4}{3},$$

a contradiction.

References

- Artem Chernikov, Ehud Hrushovski, Alex Kruckman, Krzysztof Krupinski, Slavko Moconja, Anand Pillay, Nicholas Ramsey "Invariant measures in simple and in small theories", Preprint (arXiv:2105.07281)
- Artem Chernikov "Model theory, Keisler measures and groups", The Bulletin of Symbolic Logic, 24(3), 336-339 (2018)