Recognizing groups and fields in Erdős geometry and model theory

Artem Chernikov

UCLA

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The \textit{trichotomy principle} in model theory: in a sufficiently tame context (certain strongly minimal, $o$-minimal), every structure is either “trivial”, or essentially a vector space (“modular”), or interprets a field.

Asymptotic sizes of the intersections of definable sets with finite grids in certain model-theoretically tame contexts reflect the trichotomy principle, and detect presence of algebraic structures (groups, fields).

Instances of this principle are well-known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures. Here we discuss “inverse” theorems which show this is the only way.
Sum-product and expander polynomials

- [Erdős, Szemerédi’83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,
  $$\max \{ |A + A|, |A \cdot A| \} = \Omega \left( |A|^{1+c} \right).$$

- [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any $\varepsilon$).

- [Elekes, Rónyai’00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree $d$, then for all $A, B \subseteq \mathbb{R}$,
  $$|f(A \times B)| = \Omega_d \left( n^{\frac{4}{3}} \right),$$

  unless $f$ is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials $g, h, i$. 
Elekes-Szabó theorem

[Elekes-Szabó’12] provide a conceptual generalization: for any algebraic surface $R(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:

1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq \mathbb{R}$ we have

   $$|R \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}).$$

2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \to V$ such that

   $$\pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \iff R(x_1, x_2, x_3)$$

   for all $x_i \in U_i$. 
Generalizations of the Elekes-Szabó theorem

Let $R \subseteq X_1 \times \ldots \times X_r$ be an algebraic surface (or just a definable set) with finite-to-one projection onto any $r - 1$ coordinates and $\dim(X_i) = m$.

1. [Elekes, Szabó’12] $r = 3$, $m$ arbitrary over $\mathbb{C}$ (only count on grids in general position, correspondence with a complex algebraic group of dimension $m$);
2. [Raz, Sharir, de Zeeuw’18] $r = 4$, $m = 1$ over $\mathbb{C}$;
3. [Raz, Shem-Tov’18] $m = 1$, $R$ of the form $f(x_1, \ldots, x_{r-1}) = x_r$ for any $r$ over $\mathbb{C}$.
4. [Hrushovski’13] Pseudofinite dimension, modularity
5. [Bays, Breuillard’18] $r$ and $m$ arbitrary over $\mathbb{C}$, recognized that the arising groups are abelian (no bounds on $\gamma$);
6. Related work: [Raz, Sharir, de Zeeuw’15], [Wang’15]; [Bukh, Tsimmerman’ 12], [Tao’12]; [Jing, Roy, Tran’19].
7. [C., Peterzil, Starchenko] Any $r$ and $m$, any o-minimal structure or stable with a distal expansion and explicit bounds on $\gamma$. A special case:
One-dimensional \( o \)-minimal case

Theorem (C., Peterzil, Starchenko)

Assume \( r \geq 3 \), \( \mathcal{M} \) is an \( o \)-minimal expansion of \( \mathbb{R} \) and \( R \subseteq \mathbb{R}^r \) is definable, such that the projection of \( R \) to any \( r-1 \) coordinates is finite-to-one. Then exactly one of the following holds.

1. For any finite \( A_i \subseteq \mathbb{R} \), \( i \in [r] \), we have

\[
|R \cap (A_1 \times \ldots \times A_r)| = O_{\mathcal{M}} \left(n^{r-1-\gamma}\right),
\]

where \( \gamma = \frac{1}{3} \) if \( r \geq 4 \), and \( \gamma = \frac{1}{6} \) if \( r = 3 \).

2. There exist open sets \( U_i \subseteq \mathbb{R} \), \( i \in [r] \), an open set \( V \subseteq \mathbb{R} \) containing 0, and homeomorphisms \( \pi_i : U_i \to V \) such that

\[
\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \iff R(x_1, \ldots, x_r)
\]

for all \( x_i \in U_i \), \( i \in [r] \).
General o-minimal case

Theorem (C., Peterzil, Starchenko)

Let $\mathcal{M}$ be an o-minimal expansion of $\mathbb{R}$. Assume $r \geq 3$, $R \subseteq X_1 \times \cdots \times X_r$ are definable with $\dim (X_i) = m$, and the projection of $R$ to any $r - 1$ coordinates is finite-to-one. Then exactly one of the following holds.

1. For any finite $A_i \subseteq X_i$ in general position, $i \in [r]$, we have

$$|R \cap (A_1 \times \cdots \times A_r)| = O_R \left( n^{r-1-\gamma} \right),$$

for $\gamma = \frac{1}{8m-5}$ if $s \geq 4$, and $\gamma = \frac{1}{16m-10}$ if $s = 3$.

2. There exist definable relatively open sets $U_i \subseteq X_i$, $i \in [s]$, an abelian Lie group $(G, +)$ of dimension $m$ and an open neighborhood $V \subseteq G$ of 0, and definable homeomorphisms $\pi_i : U_i \to V$, $i \in [s]$, such that for all $x_i \in U_i$, $i \in [s]$

$$\pi_1(x_1) + \cdots + \pi_s(x_s) = 0 \Leftrightarrow R(x_1, \ldots, x_s).$$
Remarks

1. If $\mathcal{M}$ is o-minimal but is not elementarily equivalent to an expansion of $\mathbb{R}$ — only get correspondence with a type-definable group.

2. One ingredient — “Szémeredi-Trotter”-style bounds in o-minimal, and more generally distal structures.

3. Another — a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a “generic chunk”, along with a purely combinatorial version.
First ingredient: Recognizing groups, 1

1. Assume that \((G, +, 0)\) is an abelian group, and consider the \(r\)-ary relation \(R \subseteq \prod_{i \in [r]} G\) given by \(x_1 + \ldots + x_r = 0\).

2. Then \(R\) is easily seen to satisfy the following two properties, for any permutation of the variables of \(R\):

\[
\forall x_1, \ldots, \forall x_{r-1} \exists! x_r R(x_1, \ldots, x_r), \tag{P1}
\]

\[
\forall x_1, x_2 \forall y_3, \ldots y_r \forall y'_3, \ldots, y'_r \left( R(\bar{x}, \bar{y}) \land R(\bar{x}, \bar{y}') \rightarrow \left( \forall x'_1, x'_2 R(\bar{x}', \bar{y}) \leftrightarrow R(\bar{x}', \bar{y}'))) \right). \tag{P2}
\]

We show a converse, assuming \(r \geq 4\):
Recognizing groups, 2

Theorem (C., Peterzil, Starchenko)

Assume \( r \in \mathbb{N}_{\geq 4}, X_1, \ldots, X_r \) and \( R \subseteq \prod_{i \in [r]} X_i \) are sets, so that \( R \) satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group \((G, +, 0_G)\) and bijections \( \pi_i : X_i \rightarrow G \) such that for every \((a_1, \ldots, a_r) \in \prod_{i \in [r]} X_i\) we have

\[
R(a_1, \ldots, a_r) \iff \pi_1(a_1) + \ldots + \pi_r(a_r) = 0_G.
\]

- If \( X_1 = \ldots = X_r \), property (P1) is equivalent to saying that the relation \( R \) is an \((r-1)\)-dimensional permutation on the set \( X_1 \), or a Latin \((r-1)\)-hypercube, as studied by Linial and Luria. Thus the condition (P2) characterizes, for \( r \geq 3 \), those Latin \( r \)-hypercubes that are given by the relation “\( x_1 + \ldots + x_{r-1} = x_r \)” in an abelian group.

- If \( R \) is definable and \( X_i \) are type-definable in a (saturated) \( M \), then \( G \) is type-definable and \( \pi_i \) are relatively definable in \( M \).
Recognizing groups in the stable case

- In the stable version of our theorem, we only get “generic correspondence” with a type-definable group.
- An $r$-gon is a tuple $a_1, \ldots, a_r$ such that any $r - 1$ of its elements are (forking-)independent, and any element in it is in the algebraic closure of the other ones.
- An $r$-gon is abelian if, after any permutation of its elements, we have $a_1 a_2 \downarrow_{\text{acl}(a_1 a_2) \cap \text{acl}(a_3 \ldots a_r)} a_3 \ldots a_r$.
- If $(G, \cdot)$ is a type-definable abelian group, $g_1, \ldots, g_{r-1}$ are independent generics in $G$ and $g_r := g_1 \cdot \ldots \cdot g_{r-1}$, then $g_1, \ldots, g_r$ is an abelian $r$-gon (associated to $G$).
- Conversely,

Theorem (C., Peterzil, Starchenko; independently Hrushovski)

Let $r \geq 4$ and $a_1, \ldots, a_r$ be an abelian $r$-gon. Then there is a type-definable (in $\mathcal{M}^{eq}$) connected abelian group $(G, \cdot)$ and an abelian $r$-gon $g_1, \ldots, g_s$ associated to $G$, such that after a base change each $g_i$ is interalgebraic with $a_i$. 
Second ingredient: distality

Definition
A structure $\mathcal{M}$ is *distal* if and only if for every definable family $\{\varphi(x, b) : b \in M_y\}$ of subsets of $M_x$ there is a definable family $\{\theta(x, c) : c \in M^k_y\}$ such that for every $a \in M_x$ and every finite set $B \subset M_y$ there is some $c \in B^k$ such that:

- $a \models \theta(x, c)$;
- $\theta(x, c) \vdash \text{tp}_{\varphi}(a/B)$, that is for every $a' \models \theta(x, c)$ and $b \in B$ we have $a' \models \phi(x, b) \iff a \models \phi(x, b)$.
Examples of distal structures

- $\mathcal{M}$ distal $\implies \mathcal{M}$ is NIP, unstable.
- Examples of distal structures: (weakly) o-minimal structures, various valued fields of char 0 (e.g. $\mathbb{Q}_p$, RCVF, the valued differential field of transseries).
- Stable structures with distal expansions: $\text{ACF}_0$, $\text{DCF}_{0,m}$, CCM, abelian groups, Hrushovski constructions*.
- Stable structures without distal expansions: $\text{ACF}_p$ [C., Starchenko’15], a disjoint union of finite expander graphs (e.g. Ramanujan graphs) of growing degree and expansion [Jiang, Nesetril, Ossona de Mendez, Siebertz’20].
- **Problem.** Do non-abelian free groups have distal expansions?
Number of edges in a $K_{k,\ldots,k}$-free hypergraph

- The following fact is due to [Kővári, Sós, Turán’54] for $r = 2$ and [Erdős’64] for general $r$.

Fact (The Basic Bound)

If $H$ is a $K_{k,\ldots,k}$-free $r$-hypergraph then $|E| = O_{r,k} \left(n^r - \frac{1}{k^r-1}\right)$.

- So the exponent is slightly better than the maximal possible $r$ (we have $n^r$ edges in $K_{n,\ldots,n}$). A probabilistic construction in [Erdős’64] shows that it cannot be substantially improved.
Bounds for graphs definable in distal structures

Generalizing [Fox, Pach, Sheffer, Suk, Zahl’15] in the semialgebraic case, we have:

Fact (C., Galvin, Starchenko’16)

Let $\mathcal{M}$ be a distal structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists some $\varepsilon = \varepsilon(R, k) > 0$ such that for any $A_1 \subseteq_n M_{x_1}, A_2 \subseteq_n M_{x_2}$, if $E := R \cap (A_1 \times A_2)$ is $K_{k,k}$-free then $|E| = O_{R,k}(n^{t-\varepsilon})$, where $t$ is the exponent given by the Basic Bound for arbitrary graphs.

In fact, $\varepsilon$ is given in terms of $k$ and the size of the smallest distal cell decomposition for $R$.

E.g. if $R \subseteq M^2 \times M^2$ for an o-minimal $\mathcal{M}$, then $t - \varepsilon = \frac{4}{3}$ ([C., Galvin, Starchenko’16]; independently, [Basu, Raz’16]).
Recognizing fields

For the semialgebraic \( K_{2,2}\)-free point-line incidence relation \( R = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1x_1 + y_2\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \) we have the (optimal) lower bound \(|R \cap (V_1 \times V_2)| = \Omega(n^{4/3})\).

To define it we use both addition and multiplication, i.e. the field structure.

This is not a coincidence — any non-trivial lower bound on the Zarankiewicz exponent of \( R \) allows to recover a field from it:

**Theorem (Basit, C., Starchenko, Tao, Tran)**

*Assume that \( \mathcal{M} = (M, <, \ldots) \) is \( o \)-minimal and \( R \subseteq M_{x_1} \times \ldots \times M_{x_r} \) is a definable relation which is \( K_{k,\ldots,k} \)-free, but \(|R \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})\) for \( V_i \subseteq_n M_{x_i} \). Then a real closed field is definable in the first-order structure \((M, <, R)\).*
Ingredients

- An (almost) optimal bound on the number of edges in $K_{k,...,k}$-free hypergraphs definable in locally modular $o$-minimal expansions of groups, so e.g. for semilinear ($= \text{definable in } (\mathbb{R}, <, +)$) hypergraphs.

- The trichotomy theorem for $o$-minimal structures [Peterzil, Starchenko’98].
A matroid associated to an o-minimal structure

- Given a structure $M$, $A \subseteq M$ and a finite tuple $a$ in $M$, $a \in \text{acl}(A)$ if it belongs to some finite $A$-definable subset of $M^{\mid a\mid}$ (this generalizes linear span in vector spaces and algebraic closure in fields).
- $\dim(a/A)$ is the minimal cardinality of a subtuple $a'$ of $a$ so that $\text{acl}(a \cup A) = \text{acl}(a' \cup A)$ (in an algebraically closed field, this is just the transcendence degree of $a$ over the field generated by $A$).
- Given a finite tuple $a$ and sets $C, B \subseteq M$, we write $a \downarrow_C B$ to denote that $\dim(a/BC) = \dim(a/C)$.
- In an o-minimal structure, $\downarrow$ is a well-behaved notion of independence defining a matroid.
Local modularity

An $o$-minimal structure is (weakly) locally modular if for any small subsets $A, B \subseteq M \models T$ there exists some small set $C \downarrow_{\emptyset} AB$ such that $A \downarrow_{\text{acl}(AC) \cap \text{acl}(BC)} B$.

Intuition: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.

In particular, an $o$-minimal structure is locally modular if and only if any normal interpretable family of plane curves in $T$ has dimension $\leq 1$. 
Theorem (Basit, C., Starchenko, Tao, Tran)

Let \( \mathcal{M} \) be an o-minimal locally modular expansion of a group and \( Q \) a definable relation of arity \( r \geq 2 \). Then for any \( \varepsilon > 0 \) and any \( V_i \) with \( |V_i| = n \) such that \( E := Q \cap V_1 \times \ldots \times V_r \) is \( K_{k,\ldots,k} \)-free, we have

\[
|E| = O_{Q,k,\varepsilon} \left( n^{r-1+\varepsilon} \right).
\]

Moreover, if \( Q \) itself is \( K_{k,\ldots,k} \)-free, then for any \( V_i \) with \( |V_i| = n \) we have

\[
|E| = O_{Q} \left( n^{r-1} \right).
\]
Recovering a field in the o-minimal case

Fact (Peterzil, Starchenko’98)

Let $\mathcal{M}$ be an o-minimal (saturated) structure. TFAE:

- $\mathcal{M}$ is not locally modular;
- there exists a real closed field definable in $\mathcal{M}$.

- [Marker, Peterzil, Pillay’92] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\cdot \upharpoonright_{[0,1]^2}$ is definable in $(\mathbb{R}, <, +, X)$. In particular, it is not locally modular.

- Combining this with the optimal bound in the locally modular case, we get the result.

- Problem: is it possible to establish a more direct correspondence between the relation with many edges and the point-line incidence relation in a field?
An application to incidences with polytopes

Applying with $r = 2$ we get the following:

Corollary

For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.
Let $d \in \mathbb{N}$ and $H_1, \ldots, H_q \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_q$.
For any set $V_1$ of $n_1$ points in $\mathbb{R}^d$ and any set $V_2$ of $n_2$ polytopes in $\mathcal{F}$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $\alpha n (\log n)^q$ incidences.

In particular (this corollary was obtained independently by [Tomon, Zakharov]):

Corollary

For any set $V_1$ of $n_1$ points and any set $V_2$ of $n_2$ (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k} \left(n(\log n)^{2d}\right)$ incidences.
Dyadic rectangles and a lower bound

- Is the logarithmic factor necessary?
- We focus on the simplest case of incidences with rectangles with axis-parallel sides in $\mathbb{R}^2$. The previous corollary gives the bound $O_{d,k}(n(\log n)^4)$.
- A box is dyadic if it is the direct products of intervals of the form $[s2^t, (s+1)2^t)$ for some integers $s, t$.
- Using a different argument, restricting to dyadic boxes we get a stronger upper bound $O\left(n\frac{\log n_1}{\log \log n_1}\right)$, and give a construction showing a matching lower bound (up to a constant).
- [Tomon, Zakharov] use our construction to disprove a conjecture of Alon, Basavaraju, Chandran, Mathew, and Rajendraaprasad regarding the maximal possible number of edges in a graph of bounded separation dimension.

Problem

What is the optimal bound on the power of $\log n$? In particular, does it have to grow with the dimension $d$?
Thank you!

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