Tame definable topological dynamics

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- Joint work with Pierre Simon, continues previous work with Anand Pillay and Pierre Simon.

Setting

- T is a complete first-order theory in a language L, countable for simplicity.
- $\mathbb{M} \models T$ — a monster model, $\kappa(\mathbb{M})$-saturated for some sufficiently large strong limit cardinal $\kappa(\mathbb{M})$.
- G — a definable group (over $\emptyset$ for simplicity).
- As usual, for any set A we denote by $S_x(A)$ the (compact, Hausdorff) space of types (in the variable x) over A and by $S_G(A) \subseteq S_x(A)$ the space of types in G.
  $\mathrm{Def}_x(A)$ denotes the boolean algebra of A-definable subsets of $\mathbb{M}$.
- G acts naturally on $S_G(\mathbb{M})$ by homeomorphisms:
  for $a \models p(x) \in S_G(\mathbb{M})$ and $g \in G(\mathbb{M})$,
  $g \cdot p = \mathrm{tp}(g \cdot a) = \left\{\phi(x) \in L(\mathbb{M}) : \phi\left(g^{-1} \cdot x\right) \in p\right\}$.

VC-dimension

- Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of subsets of a set S.
- For a set $B \subseteq S$, let $\mathcal{F} \cap B = \{X_a \cap B : a \in A\}$.
- We say that $B \subseteq S$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- Let $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subset S, |B| = n\}$.
- If the VC dimension of $\mathcal{F}$ is infinite, then $\pi_{\mathcal{F}}(n) = 2^n$ for all n. However,

Fact

[Sauer-Shelah lemma] If $\mathcal{F}$ has VC dimension d, then $\pi_{\mathcal{F}}(n) = O(n^d)$.

- Computational learning theory, probability/combinatorics, functional analysis, model theory...

NIP theories

- A formula $\phi(x, y)$ (where x, y are tuples of variables) is NIP if the family $\mathcal{F}_\phi = \{\phi(x, a) : a \in \mathbb{M}\}$ has finite VC-dimension.
- T is NIP if it implies that every formula $\phi(x, y) \in L$ is NIP.

Fact

[Shelah] T is NIP *iff every formula $\phi(x, y)$ with $|x| = 1$ is* NIP*.*

- Examples of NIP theories:
  - stable theories (e.g. modules, algebraically / separably / differentially closed fields, free groups by Sela),
  - *o*-minimal theories (e.g. real closed fields with exponentiation),
  - ordered abelian groups,
  - algebraically closed valued fields, *p*-adics.
- Non-examples: the theory of the random graph, pseudo-finite fields, ...

Model-theoretic connected components

Let A be a small subset of $\mathbb{M}$. We define:

- $G_A^0 = \bigcap \{H \leq G : H$ is A-definable, of finite index$\}$.
- $G_A^{00} =$
  $\bigcap \{H \leq G : H$ is type-definable over A, of bounded index, i.e. $< \kappa (\mathbb{M}$
- $G_A^{\infty} =$
  $\bigcap \{H \leq G : H$ is $\mathrm{Aut}(\mathbb{M} / A)$-invariant, of bounded index$\}$.
- Of course $G_A^0 \supseteq G_A^{00} \supseteq G_A^{\infty}$, and in general all these subgroups get smaller as A grows.

Connected components in NIP

Fact

Let T be NIP. Then for every small set A we have:

- *[Baldwin-Saxl] $G^0_\emptyset = G^0_A$,*
- *[Shelah] $G^{00}_\emptyset = G^{00}_A$,*
- *[Shelah for abelian groups, Gismatullin in general] $G^\infty_\emptyset = G^\infty_A$.*
- *All these are normal $\mathrm{Aut}(\mathbb{M})$-invariant subgroups of G of finite (resp. bounded) index. We will be omitting $\emptyset$ in the subscript.*

Example

[Conversano, Pillay] There are NIP groups in which $G^{00} \neq G^\infty$ (G is a saturated elementary extension of $\widetilde{SL(2,\mathbb{R})}$, the universal cover of $SL(2,\mathbb{R})$, in the language of groups. G is not actually denable in an o-minimal structure, but one can give another closely related example which is).

The logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map.
- We endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.
- With this topology, G/G^{00} is a compact topological group.
- In particular, there is a normalized left-invariant Haar probability measure h_0 on it.

Examples

1. If $G^0 = G^{00}$ (e.g. G is a stable group), then G/G^{00} is a profinite group: it is the inverse image of the groups G/H, where H ranges over all definable subgroups of finite index.
2. If $G = \mathrm{SO}(2, \mathcal{R})$ is the circle group defined in a real closed field $\mathcal{R}$, then G^{00} is the set of infinitesimal elements of G and G/G^{00} is canonically isomorphic to the standard circle group $SO(2, \mathbb{R})$.
3. More generally, if G is any definably compact group defined in an o-minimal expansion of a field, then G/G^{00} is a compact Lie group. This is part of the content of Pillay's conjecture (now a theorem).
4. This does not hold any more if G is a non-compact Lie group. For example if $G = (\mathbb{R}, +)$, then $G^{00} = G$ and G/G^{00} is trivial.

Keisler measures

- A *Keisler measure* μ over a set of parameters $A \subseteq \mathbb{M}$ is a finitely additive probability measure on the boolean algebra $\mathrm{Def}_x(A)$.
- $S(\mu)$ denotes the support of μ, i.e. the closed subset of $S_x(A)$ such that if $p \in S(\mu)$, then $\mu(\phi(x)) > 0$ for all $\phi(x) \in p$.
- Let $\mathfrak{M}_x(A)$ be the space of Keisler measures over A. It can be naturally viewed as a closed subset of $[0,1]^{L(A)}$ with the product topology, so $\mathfrak{M}_x(A)$ is compact. Every type can be associated with a Dirac measure concentrated on it.

Fact

There is a natural bijection $\{$*Keisler measures over* $A\} \leftrightarrow$ $\{$*Regular Borel probability measures on* $S(A)\}$.

- We will use this equivalence freely and will just say "measure".

The weak law of large numbers

- Let (X, μ) be a probability space.
- Given a set $S \subseteq X$ and $x_1, \ldots, x_n \in X$, we define $\mathrm{Av}\left(x_1, \ldots, x_n; S\right) = \frac{1}{n} \left|S \cap \left\{x_1, \ldots, x_n\right\}\right|$.
- For $n \in \omega$, let μ^n be the product measure on X^n.

Fact

(Weak law of large numbers) Let $S \subseteq X$ be measurable and fix $\varepsilon > 0$. Then for any $n \in \omega$ we have:

$$\mu^n\left(\bar{x} \in X^n : \left|\mathrm{Av}\left(x_1, \ldots, x_n; S\right) - \mu\left(S\right)\right| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2}.$$

A uniform version for families of finite VC dimension

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a family of measurable subsets of X of finite VC-dimension such that:

1. *for each n, the function $f_n(x_1, \ldots, x_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)|$ is a measurable function from X^n to $\mathbb{R}$;*
2. *for each n, the function $g_n(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mathrm{Av}(x'_1, \ldots, x'_n; S)|$ from X^{2n} to $\mathbb{R}$ is measurable.*

Then for every $\varepsilon > 0$ and $n \in \omega$ we have:

$$\mu^n\left(\sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)| > \varepsilon\right) \leq 8\pi_{\mathcal{F}}(n) \exp\left(-\frac{n\varepsilon^2}{32}\right).$$

(recall that $\pi_{\mathcal{F}}(n)$ is bounded by a polynomial by Sauer-Shelah).

Approximating

- In particular this implies that in NIP measures can be approximated by the averages of types:

Corollary

(*) [Hrushovski, Pillay] Let T be NIP, $\mu \in \mathfrak{M}_x(A)$, $\phi(x, y) \in L$ *and* $\varepsilon > 0$ *arbitrary. Then there are some* $p_0, \ldots, p_{n-1} \in S(\mu)$ *such that* $\mu(\phi(x, a)) \approx^{\varepsilon} \mathrm{Av}(p_0, \ldots, p_{n-1}; \phi(x, a))$ *for all* $a \in \mathbb{M}$.

Definably amenable groups

Definition

A definable group G is *definably amenable* if there is a global (left) G-invariant Keisler measure on G.

- If G is definably amenable, then it also admits a global measure which is right-invariant $(\nu(\phi(x)) = \mu(\phi(x^{-1})))$.
- If for some model M there is a left-invariant Keisler measure μ_0 on M-definable sets, then G is definably amenable.

Examples of definably amenable groups

1. If for some model M, the group $G(M)$ is amenable as a discrete group, then G is definably amenable.
2. If G admits a left-invariant type, that is a global type p such that $g \cdot p = p$ for all $g \in G$, then it is definably amenable. Such groups are called *definably extremely amenable*.
3. Suppose T has a model M such that G is defined over M and $G(M)$ has a compact (Hausdorff) group topology such that every definable subset of G is Haar measurable. Then G is definably amenable. (e.g. let $G(\mathbb{R})$ be a compact Lie group, seen as a definable group in RCF. Then G is definably amenable)
4. In particular, the group $SO_3(\mathbb{R})$ *is* definably amenable, but it is not amenable (Banach-Tarski paradox).
5. More generally, definably compact groups in o-minimal structures are definably amenable.

Examples of definably amenable groups

Examples

1. Any stable groups is definably amenable. In particular the free group F_2 is known by the work of Sela to be stable as a pure group, and hence is definably amenable.
2. Any pseudo-finite group is definably amenable.
3. If K is an algebraically closed valued field or a real closed field and $n > 1$, then $SL(n, K)$ is not definably amenable.

Problem

- **Problem**. Classify all G-invariant measures in a definably amenable group (to some extent)?
- The set of measures on $S(\mathbb{M})$ can be naturally viewed as a subset of $C^*(S(\mathbb{M}))$, the dual space of the topological vector space of continuous functions on $S(\mathbb{M})$, with the weak* topology of pointwise convergence (i.e. $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for all $f \in C(S(\mathbb{M}))$). One can check that this topology coincides with the topology on the space of $\mathfrak{M}(\mathbb{M})$ that we had introduced before.
- The set of G-invariant measures is a compact convex subset, and extreme points of this set are called *ergodic* measures.
- Using Choquet theory, one can represent arbitrary measures as integral averages over extreme points.
- We will characterize ergodic measures on G as liftings of the Haar measure on G/G^{00} w.r.t. certain "generic" types.

Invariant types

Definition

1. A global type $p \in S_x(\mathbb{M})$ is *invariant over a small set* A if $p = \sigma p$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, where $\sigma p = \{\phi(x, \sigma(a)) : \phi(x, a) \in p\}$.
2. A global type $p \in S_x(\mathbb{M})$ is invariant if it is invariant over some small model M.

- Every definable type is invariant. In fact, a weak converse is true in NIP:

Fact

1. *[Hrushovski, Pillay] If T is* NIP *and $p \in S_x(\mathbb{M})$ is invariant over M, then it is Borel-definable over M, more precisely for every $\phi(x, y) \in L$ the set $\{a \in \mathbb{M} : \phi(x, a) \in p\}$ is defined by a finite boolean combination of type-definable sets over M.*
2. *[Shelah] If T is* NIP *and M is a small model, then there are at most $2^{|M|}$ global M-invariant types.*

Strongly f-generic types

- Now we also have a definable group G acting on types.

Definition

A global type $p \in S_x(\mathbb{M})$ is *strongly f-generic* if there is a small model M such that $g \cdot p$ is invariant over M for all $g \in G(\mathbb{M})$.

Fact

[Hrushovski, Pillay]

1. *An* NIP *group is definably amenable if and only if there is a strongly f-generic type.*
2. *If $p \in S_G(\mathbb{M})$ is strongly f-generic then* $\mathrm{Stab}(p) = G^{00} = G^{\infty}$ *(where* $\mathrm{Stab}(p) = \{g \in G : gp = p\}$*).*

f-generic types

Definition

A global type $p \in S_x(\mathbb{M})$ is *f-generic* if for every $\phi(x) \in p$ and some/any small model M such that $\phi(x) \in L(M)$ and any $g \in G(\mathbb{M})$, $g \cdot \phi(x)$ contains a global M-invariant type.

Theorem

Let G be an NIP *group, and $p \in S_G(\mathbb{M})$.*

1. *G is definably amenable if and only if it has a bounded orbit (i.e. exist $p \in S_G(\mathbb{M})$ s.t. $|Gp| < \kappa(\mathbb{M})$).*
2. *If G is definably amenable, then p is f-generic iff it is G^{00}-invariant iff* $\mathrm{Stab}(p)$ *has bounded index in G iff the orbit of p is bounded.*

- (1) confirms a conjecture of Petrykowski in the case of NIP theories (it was previously known in the o-minimal case [Conversano-Pillay]).
- Our proof uses the theory of forking over models in NIP from [Ch., Kaplan] (I'll say more later in the talk).

f-generic vs strongly f-generic

- Are the notions of f-generic and strongly f-generic different?
- **Proposition.** $p \in S(\mathbb{M})$ is strongly f-generic iff it is f-generic and invariant over some small model M.

Example

There are f-generic types which are not strongly f-generic. Let $\mathcal{R}$ be a saturated model of RCF, and let $G = (R^2, +)$. Let $p(x)$ denote the definable 1-type at $+\infty$ and $q(y)$ a global 1-type which is not invariant over any small model (hence corresponds to a cut of maximal cofinality from both sides). Then p and q are weakly orthogonal types, i.e. $p(x) \cup q(y)$ determines a complete type. Let $(a, b) \models p(x) \cup q(y)$ and consider $r = \mathrm{tp}(a, a \cdot b/\mathcal{R})$. Then r is a G-invariant type which is not invariant over any small model.

Lifting measures from G/G^{00}

- We explain the connection between G-invariant measures and f-generic types.
- Let $p \in S_G(\mathbb{M})$ be f-generic (so in particular gp is G^{00}-invariant for all $g \in G$).
- Let $A_{\phi,p} = \{\bar{g} \in G/G^{00} : \phi(x) \in g \cdot p\}$.
- **Claim.** $A_{\phi,p}$ is a measurable subset of G/G^{00} (using Borel-definability of invariant types in NIP).

Definition

For $\phi(x) \in L(\mathbb{M})$, we define $\mu_p(\phi(x)) = h_0(A_{\phi,p})$.

- The measure μ_p is G-invariant and $\mu_{g \cdot p} = \mu_p$ for any $g \in G$.

Properties of μ_p's

- **Lemma.** For a fixed formula $\phi(x, y)$, let $\mathbf{A}_\phi$ be the family of all $A_{\phi(x,b),p}$ where b varies over $\mathbb{M}$ and p varies over all f-generic types. Then $\mathbf{A}_\phi$ has finite VC-dimension.
- **Corollary.** For fixed $\phi(x) \in L(\mathbb{M})$ and an f-generic $p \in S_x(\mathbb{M})$, the family $\mathcal{F} = \left\{ g \cdot A_{\phi,p} : g \in G/G^{00} \right\}$ has finite VC-dimension (as changing the formula we can assume that every translate of ϕ is an instance of ϕ).

Lemma (**). For any $\phi(x) \in L$, $\varepsilon > 0$ and a *finite* collection of f-generic types $(p_i)_{i<n}$ there are some $g_0, \ldots, g_{m-1} \in G$ such that for any $g \in G$ and $i \in \omega$ we have
$\mu_{p_i}(g \cdot \phi(x)) \approx^{\varepsilon} \mathrm{Av}(g_j \cdot g \cdot \phi(x) \in p_i)$.

Proof.

Enough to be able to apply the VC-theorem to the family $\mathcal{F}$. It has finite VC-dimension by the previous corollary, we have to check that f_n, g_n are measurable for all $n \in \omega$. Using invariance of h_0 this can be reduced to checking that certain analytic sets are measurable. As L is countable, G/G^{00} is a Polish space (the logic topology can be computed over a fixed countable model). Luckily, analytic sets in Polish spaces are universally measurable (follows from the projective determinacy for analytic sets). □

- Remark. In fact the proof shows that one can replace finite by *countable*.

Properties of μ_p's

Proposition. Let p be an f-generic type, and assume that $q \in \overline{Gp}$. Then q is f-generic and $\mu_p = \mu_q$.

Proof.

q is f-generic as the space of f-generic types is closed. Fix some $\phi(x)$. It follows from Lemma (**) that the measure $\mu_p(\phi(x))$ is determined up to ε by knowing which cosets of $\phi(x)$ belong to p. These cosets are the same for both types p and q by topological considerations on $S_x(\mathbb{M})$. $\square$

- It follows that for a given G-invariant measure μ, the set of f-generic types p for which $\mu_p = \mu$ is closed.

Properties of μ_p's

Proposition. Let p be f-generic. Then for any definable set $\phi(x)$, if $\mu_p(\phi(x)) > 0$, then there is a finite union of translates of $\phi(x)$ which has μ_p-measure 1.

Proof.

Can cover the support $S(\mu_p)$ by finitely many translates using the previous lemma and compactness. □

Proposition. Let μ be G-invariant, and assume that $p \in S(\mu)$. Then p is f-generic.

Proof.

Fix $\phi(x) \in p$, let M be some small model such that ϕ is defined over M. By [Ch., Pillay, Simon], every $G(M)$-invariant measure μ on $S(M)$ extends to a global G-invariant, M-invariant measure μ' (one can take an "invariant heir" of μ). As $\mu|_M(\phi(x)) > 0$, it follows that $\phi(x) \in q$ for some $q \in S(\mu')$. But every type in the support of an M-invariant measure is M-invariant. □

Properties of μ_p's

Lemma (***). Let μ be G-invariant. Then for any $\varepsilon > 0$ and $\phi(x, y)$, there are some f-generic $p_0, \ldots, p_{n-1}$ such that $\mu(\phi(x, b)) \approx^{\varepsilon} \mathrm{Av}(\mu_{p_i}(\phi(x, b)))$ for any $b \in \mathbb{M}$.

Proof.

- WLOG every translate of an instance of ϕ is an instance of ϕ.
- On the one hand, by Lemma (*) and G-invariance of μ there are types $p_0, \ldots, p_{n-1}$ from the support of μ such that $\mu(\phi(x, b)) \approx^{\varepsilon} \mathrm{Av}(g\phi(x, b) \in p_i)$ for any $g \in G$ and $b \in \mathbb{M}$.
- By the previous lemma p_i's are f-generic.
- On the other hand, by Lemma (**) for every $b \in \mathbb{M}$ there are some $g_0, \ldots, g_{m-1} \in G$ such that for any $i < n$, $\mu_{p_i}(\phi(x, b)) \approx^{\varepsilon} \mathrm{Av}(g_j \cdot \phi(x, b) \in p_i)$.
- Combining and re-enumerating we get that $\mu(\phi(x, b)) \approx^{2\varepsilon} \mathrm{Av}(\mu_{p_i}(\phi(x, b)))$.

□

Ergodic measures

Theorem

Global ergodic measures are exactly the measures of the form μ_p for p varying over f-generic types.

Proof: μ_p's are ergodic.

- We had defined ergodic measures as extreme points of the convex set of G-invariant measures.
- Equivalently, a G-invariant measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ is *ergodic* if $\mu(Y)$ is either 0 or 1 for every Borel set $Y \subseteq S_x(\mathbb{M})$ such that $\mu(Y \triangle gY) = 0$ for all $g \in G$.
- Fix a global f-generic type p, and for any Borel set $X \subseteq S(\mathbb{M})$ let $f_p(X) = \{g \in G/G^{00} : gp \in X\}$. Note that $f_p(X)$ is Borel. The measure μ_p defined earlier extends naturally to all Borel sets by taking $\mu_p(X) = h_0(f_p(X))$, defined this way it coincides with the usual extension of a finitely additive Keisler measure μ_p to a regular Borel measure.
- As h_0 is ergodic on G/G^{00} and $f_p(X \triangle gX) = f_p(X) \triangle g f_p(X)$, it follows that μ_p is ergodic.

Proof: μ ergodic $\Rightarrow \mu = \mu_p$ for some f-generic p

- Let μ be an ergodic measure.
- By Lemma (**) , as L is countable, μ can be written as a limit of a sequence of averages of measures of the form μ_p.
- Let S be the set of all μ_p's ocurring in this sequence, S is countable.
- It follows that $\mu \in \overline{\text{Conv}S}$, and it is still an extreme point of $\overline{\text{Conv}S}$.
- Fact [e.g. Bourbaki]. Let E be a real, locally convex, linear Hausdorff space, and C a compact convex subset of E, $S \subseteq C$. Then $C = \overline{\text{Conv}S}$ iff $\overline{S}$ includes all extreme points of C.
- Then actually $\mu \in \overline{S}$.
- It remains to check that if p is the limit of a *countable* set of p_i's along some ultrafilter $\mathcal{U}$, then also the μ_{p_i}'s converge to μ_p along $\mathcal{U}$. By the countable version of Lemma (*), given $\varepsilon > 0$ and $\phi(x)$, we can find $g_0, \ldots, g_{m-1} \in G$ such that $\mu_{p_i}(\phi(x)) \approx^{\varepsilon} \text{Av}(g_j \phi(x) \in p_i)$ for all $i \in \omega$. But then $\{i \in \omega : \mu_{p_i}(\phi(x)) \approx^{\varepsilon} \mu_p(\phi(x))\} \in \mathcal{U}$, so we can conclude.

Several notions of genericity

- Another basic question: when a definable set contains a "generic" type? And also what is the right definition of "generic" outside of the stable context?
- For the action of the automorphism group, i.e. whether a definable set contains an invariant type – the answer is given by the theory of forking.
- Action of a definable group G – ... as well.

Several notions of genericity

- Stable setting: a formula $\phi(x)$ is *generic* if there are finitely many elements $g_0, \ldots, g_{n-1} \in G$ such that $G = \bigcup_{i<n} g_i \cdot \phi(x)$.
- A global type $p \in S_x(\mathbb{M})$ is generic if every formula in it is generic.
- Problem: generic types need not exist in unstable groups.
- Several weakenings coming from different contexts were introduced by different people (in the definably amenable setting, and more generally).

Several notions of genericity

Theorem

Let G be definably amenable, NIP*. Then the following are equivalent*

1. $\phi(x)$ *is f-generic (i.e. belongs to an f-generic type),*
2. $\phi(x)$ *is weakly generic (i.e. exists a non-generic $\psi(x)$ such that $\phi(x) \cup \psi(x)$ is generic),*
3. $\mu(\phi(x)) > 0$ *for some G-invariant measure μ,*
4. $\mu_p(\phi(x)) > 0$ *for some ergodic measure μ_p.*

If there is a generic type, then all these notions are equivalent to "$\phi(x)$ is generic".

- **Proposition.** G admits a generic type iff it is uniquely ergodic. In this case the invariant measure is both left and right invariant.

Some comments on the proof

The key step is the following:

- **Proposition.** Let $\phi(x)$ be f-generic. Then there are some global f-generic types $p_0, \ldots, p_{n-1} \in S_G(\mathbb{M})$ such that for every $g \in G(\mathbb{M})$ we have $g\phi(x) \in p_i$ for some $i < n$.
- Our proof is a combination of some results on forking and the so-called (p, q)-theorem.

Dividing and forking

Definition

1. A formula $\phi(x, a)$ *divides* over a set A if there is a sequence $(a_i)_{i \in \omega} \in \mathbb{M}$ and $k \in \omega$ such that:
   1.1 $\text{tp}(a_i/A) = \text{tp}(a/A)$ for all $i < \omega$,
   1.2 the family $\{\phi(x, a_i)\}_{i \in \omega}$ is k-inconsistent (i.e. for every $i_0 < i_1 \ldots < i_{k-1} \in \omega$ we have $\bigcap_{i<k} \phi(x, a_i) = \emptyset$).
2. A formula $\phi(x, a)$ *forks* over A if there are finitely many $\psi_0(x, b_0), \ldots, \psi_{n-1}(x, b_{n-1})$ such that $\phi(x, a) \vdash \bigvee_{i<n} \psi_i(x, b_i)$ and each of $\psi_i(x, b_i)$ divides over A.
3. The set of formulas forking over A is an ideal in $\text{Def}(\mathbb{M})$ generated by the formulas dividing over A.

Dividing and forking

Fact

Let T be NIP, *M a small model and $\phi(x,a)$ is a formula. Then the following are equivalent:*

1. *There is a global M-invariant type $p(x)$ such that $\phi(x,a) \in p$.*
2. *$\phi(x,a)$ does not divide over M.*

- This is a combination of non-forking=invariance for global types and a theorem of [Ch.,Kaplan] on forking=dividing for formulas in NIP.
- With this fact, a formula $\phi(x)$ is f-generic iff for every M over which it is defined, and for every $g \in G(\mathbb{M})$, $g\phi(x)$ does not divide over M.

Adding G to the picture

- G is definably amenable, NIP.

Theorem

1. *Non-f-generic formulas form an ideal (in particular every f-generic formula extends to a global f-generic type by Zorn's lemma).*
2. *Moreover, this ideal is S1 in the terminology of Hrushovski: assume that $\phi(x)$ is f-generic and definable over M. Let $(g_i)_{i \in \omega}$ be an M-indiscernible sequence, then $g_0\phi(x) \wedge g_1\phi(x)$ is f-generic.*
3. *There is a form of lowness for f-genericity, i.e. for any formula $\phi(x, y) \in L(M)$, the set $B_\phi = \{b \in \mathbb{M} : \phi(x, b) \text{ is not } f\text{-generic}\}$ is type-definable over M.*

(p, q)-theorem

Definition

We say that $\mathcal{F} = \{X_a : a \in A\}$ satisfies the (p, q)-*property* if for every $A' \subseteq A$ with $|A'| \geq p$ there is some $A'' \subseteq A'$ with $|A''| \geq q$ such that $\bigcap_{a \in A''} X_a \neq \emptyset$.

Fact

[Alon, Kleitman]+[Matousek] Let $\mathcal{F}$ be a finite family of subsets of S of finite VC-dimension d. Assume that $p \geq q \gg d$. Then there is an $N = N(p, q)$ such that if $\mathcal{F}$ satisfies the (p, q)-property, then there are $b_0, \ldots, b_N \in S$ such that for every $a \in A$, $b_i \in X_a$ for some $i < N$.

- The point is that if $\phi(x)$ is f-generic, then the family $\mathcal{F} = \{g\phi(x) \cap Y : g \in G\}$ with Y the set of global f-generic types, satisfies the (p, q)-property for some p and q.