

Recognizing groups and fields in Erdős geometry and model theory

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Hypergraphs and Zarankiewicz's problem

- ▶ We fix $r \in \mathbb{N}_{\geq 2}$ and let $H = (V_1, \dots, V_r; E)$ be an r -partite and r -uniform hypergraph (or just r -hypergraph) with vertex sets V_1, \dots, V_r with $|V_i| = n_i$, (hyper-) edge set $E \subseteq \prod_{i \in [r]} V_i$, and $n = \sum_{i=1}^r n_i$ is the total number of vertices.
- ▶ When $r = 2$, we say “bipartite graph” instead of “2-hypergraph”.
- ▶ For $k \in \mathbb{N}$, let $K_{k, \dots, k}$ denote the complete r -hypergraph with each part of size k (i.e. $V_i = [k]$ and $E = \prod_{i \in [k]} V_i$).
- ▶ H is $K_{k, \dots, k}$ -free if it does not contain an isomorphic copy of $K_{k, \dots, k}$.
- ▶ Zarankiewicz's problem: for fixed r, k , what is the maximal number of edges $|E|$ in a $K_{k, \dots, k}$ -free r -hypergraph H ? (As a functions of n_1, \dots, n_r).

Number of edges in a $K_{k,\dots,k}$ -free hypergraph

- ▶ The following fact is due to [Kővári, Sós, Turán'54] for $r = 2$ and [Erdős'64] for general r .

Fact (The Basic Bound)

If H is a $K_{k,\dots,k}$ -free r -hypergraph then $|E| = O_{r,k} \left(n^{r - \frac{1}{k^{r-1}}} \right)$.

- ▶ “ $= O_{r,k}(-)$ ” means “ $\leq c \cdot -$ ” for some constant $c \in \mathbb{R}$ depending only on r and k .
- ▶ So the exponent is slightly better than the maximal possible r (we have n^r edges in $K_{n,\dots,n}$). A probabilistic construction in [Erdős'64] shows that it cannot be substantially improved (but whether it is sharp up to a constant is open).

Families of hypergraphs induced by definable relations

- ▶ Let $\mathcal{M} = (M, \dots)$ be a first-order structure in a language \mathcal{L} , and let $R \subseteq M_{x_1} \times \dots \times M_{x_r}$ be a definable relation on the product of some sorts of \mathcal{M} .
- ▶ We let \mathcal{F}_R be the family of all finite r -hypergraphs induced by R , i.e. hypergraphs of the form

$$H = (V_1, \dots, V_r; R \upharpoonright_{V_1 \times \dots \times V_r})$$

for some finite $V_i \subseteq M_{x_i}, i \in [r]$.

- ▶ **Question.** What properties of the structure \mathcal{M} are reflected by the *exponents* in Zarankiewicz-style bounds for the families of hypergraphs \mathcal{F}_R with R definable in \mathcal{M} ?
- ▶ Turns out many: NIP, distality, in certain “geometric” contexts — the trichotomy principle.

NIP graphs

Fact (Fox, Pach, Sheffer, Suk, Zahl '15)

Let \mathcal{M} be an NIP structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists $d = d(R)$ (can take $d = VC(R)$), such that for every k we have: every $K_{k,k}$ -free bipartite graph $H \in \mathcal{F}_R$ satisfies $|E| = O_{d,k} \left(n^{2 - \frac{1}{d}} \right)$.

- ▶ Conversely, independence of the bounding exponent from k implies that R is NIP:
 1. if R is not NIP, it contains every bipartite graph as an induced subgraph, and
 2. [Bohman, Keevash'10] $\forall k \geq 5$, there exists a bipartite $K_{k,k}$ -free graph with $\geq cn^{2 - \frac{2}{k+1}}$ edges.

Point-line incidences, char p

- ▶ Let $K \models \text{ACF}_p$ be an algebraically closed field of positive characteristic.
- ▶ Let $R \subseteq K^2 \times K^2$ be the (definable) incidence relation between points and lines in K^2 , i.e.

$$R(x_1, x_2; y_1, y_2) \iff x_2 = y_1 x_1 + y_2.$$

- ▶ Note that R is $K_{2,2}$ -free (there is a unique line through any two distinct points).
- ▶ Let q be a power of p , then $\mathbb{F}_q \subseteq K$ and we take $V_1 = V_2 = (\mathbb{F}_q)^2$ (i.e. the set of all points and the set of all lines in \mathbb{F}_q^2), $E = R \upharpoonright_{V_1 \times V_2}$. Then $H = (V_1, V_2; E) \in \mathcal{F}_R$.
- ▶ We have $|V_1| = |V_2| = q^2$ and $|E| = q|V_2| = q^3$.
- ▶ Let $n := q^2$, then $|V_1| = |V_2| = n$ and $|E| \geq n^{\frac{3}{2}}$ — matches the Basic Bound for $r = k = 2$.

Points-lines incidences, char 0

- ▶ On the other hand, over the reals a bound strictly better than the Basic Bound holds ($\frac{4}{3} < \frac{3}{2}$):

Fact (Szémeredi-Trotter '83)

Let $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the incidence relation between points and lines in \mathbb{R}^2 . Then every $H \in \mathcal{F}_R$ satisfies $|E| = O\left(n^{\frac{4}{3}}\right)$.

- ▶ Known to be optimal up to a constant.
- ▶ In fact, the same holds in ACF_0 :

Fact (Tóth '03)

Let $R \subseteq \mathbb{C}^2 \times \mathbb{C}^2$ be the incidence relation between points and lines in \mathbb{C}^2 . Then every $H \in \mathcal{F}_R$ satisfies $|E| = O\left(n^{\frac{4}{3}}\right)$.

- ▶ Szémeredi-Trotter theorem has numerous generalizations for semialgebraic graphs [Pach, Sharir'98], [Elekes, Szabó'12], [Fox, Pach, Sheffer, Suk, Zahl'15].
- ▶ Reason: ACF_0 is a reduct of a *distal* theory, while ACF_p is not.

Distality

Definition

A structure \mathcal{M} is *distal* if and only if for every definable family $\{\varphi(x, b) : b \in M_y\}$ of subsets of M_x there is a definable family $\{\theta(x, c) : c \in M_y^k\}$ such that for every $a \in M_x$ and every finite set $B \subset M_y$ there is some $c \in B^k$ such that:

- ▶ $a \models \theta(x, c)$;
- ▶ $\theta(x, c) \vdash \text{tp}_\varphi(a/B)$, that is for every $a' \models \theta(x, c)$ and $b \in B$ we have $a' \models \varphi(x, b) \Leftrightarrow a \models \varphi(x, b)$.

Examples of distal structures

- ▶ \mathcal{M} distal $\implies \mathcal{M}$ is NIP, unstable.
- ▶ Examples of distal structures: (weakly) σ -minimal structures, various valued fields of char 0 (e.g. \mathbb{Q}_p , RCVF, the valued differential field of transseries).
- ▶ Stable structures with distal expansions: ACF_0 , $\text{DCF}_{0,m}$, CCM, abelian groups.
- ▶ Stable structures without distal expansions: ACF_p [C., Starchenko'15], a disjoint union of finite expander graphs (e.g. Ramanujan graphs) of growing degree and expansion [Jiang, Neseiril, Ossoana de Mendez, Siebertz'20].
- ▶ **Problem.** Do non-abelian free groups have distal expansions?

Stronger bounds for hypergraphs definable in distal structures

- ▶ Generalizing [Fox, Pach, Sheffer, Suk, Zahl'15] in the semialgebraic case, we have:

Fact (C., Galvin, Starchenko'16)

Let \mathcal{M} be a distal structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists some $\varepsilon = \varepsilon(R, k) > 0$ such that every $K_{k,k}$ -free bipartite graph $H \in \mathcal{F}_R$ satisfies $|E| = O_{R,k}(n^{t-\varepsilon})$, where t is the exponent given by the Basic Bound of Erdős for arbitrary graphs.

- ▶ In fact, ε is given in terms of k and the size of the smallest distal cell decomposition for R .
- ▶ E.g. if $R \subseteq M^2 \times M^2$ for an \mathcal{o} -minimal \mathcal{M} , then $t - \varepsilon = \frac{4}{3}$ ([C., Galvin, Starchenko'16]; independently, [Basu, Raz'16]).
- ▶ Bounds for $R \subseteq M^{d_1} \times M^{d_2}$ with $\mathcal{M} \models \text{RCF}$ [Fox, Pach, Sheffer, Suk, Zahl'15]; hypergraphs [Do'18]; \mathcal{M} is \mathcal{o} -minimal [Anderson'21].

Connections to the “trichotomy principle” in model theory

- ▶ If \mathcal{M} is sufficiently tame model-theoretically (e.g. stable/geometric + distal expansion; or more concretely, ACF_0 or \mathcal{o} -minimal), the exponents in Zarankiewicz bounds of definable (hyper-)graphs appear to reflect the trichotomy principle, and detect presence of algebraic structures (groups, fields).
- ▶ Instances of this principle are also known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures.
- ▶ We discuss two results demonstrating that they *have* to come from algebraic structures in certain cases, focusing here on the \mathcal{o} -minimal case.

Elekes-Szabó theorem, 1

- ▶ [Erdős, Szemerédi'83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,

$$\max \{|A + A|, |A \cdot A|\} = \Omega(|A|^{1+c}).$$

- ▶ [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any ε).
- ▶ [Elekes, Rónyai'00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree d , then for all $A, B \subseteq_n \mathbb{R}$,

$$|f(A \times B)| = \Omega_d \left(n^{\frac{4}{3}} \right),$$

unless f is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials g, h, i .

Elekes-Szabó theorem, 2

- [Elekes-Szabó'12] provide a conceptual generalization: for any algebraic surface $R(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:

1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq_n \mathbb{R}$ we have

$$|R \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}).$$

2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \rightarrow V$ such that

$$\pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \Leftrightarrow R(x_1, x_2, x_3)$$

for all $x_i \in U_i$.

Generalizations of the Elekes-Szabó theorem

Let $R \subseteq X_1 \times \dots \times X_r$ be an algebraic surface (or just a definable set) with finite-to-one projection onto any $r - 1$ coordinates and $\dim(X_i) = m$.

1. [Elekes, Szabó'12] $r = 3$, m arbitrary over \mathbb{C} (only count on grids in *general position*, correspondence with a complex algebraic group of dimension m);
2. [Raz, Sharir, de Zeeuw'18] $r = 4$, $m = 1$ over \mathbb{C} ;
3. [Raz, Shem-Tov'18] $m = 1$, R of the form $f(x_1, \dots, x_{r-1}) = x_r$ for any r over \mathbb{C} .
4. [Bays, Breuillard'18] r and m arbitrary over \mathbb{C} , recognized that the arising groups are abelian (however no bounds on γ);
5. Related work: [Raz, Sharir, de Zeeuw'15], [Wang'15]; [Bukh, Tsimmerman' 12], [Tao'12]; [Hrushovski'13]; [Jing, Roy, Tran'19].
6. [C., Peterzil, Starchenko] Any r and m , any \mathcal{o} -minimal structure (or stable with a distal expansion) and explicit bounds on γ . A special case:

One-dimensional o-minimal case

Theorem (C., Peterzil, Starchenko)

Assume $r \geq 3$, \mathcal{M} is an o-minimal expansion of \mathbb{R} and $R \subseteq \mathbb{R}^r$ is definable, such that the projection of R to any $r - 1$ coordinates is finite-to-one. Then exactly one of the following holds.

1. For any finite $A_i \subseteq_n \mathbb{R}$, $i \in [r]$, we have

$$|R \cap (A_1 \times \dots \times A_r)| = O_R(n^{r-1-\gamma}),$$

where $\gamma = \frac{1}{3}$ if $r \geq 4$, and $\gamma = \frac{1}{6}$ if $r = 3$.

2. There exist open sets $U_i \subseteq \mathbb{R}$, $i \in [r]$, an open set $V \subseteq \mathbb{R}$ containing 0, and homeomorphisms $\pi_i : U_i \rightarrow V$ such that

$$\pi_1(x_1) + \dots + \pi_r(x_r) = 0 \Leftrightarrow R(x_1, \dots, x_r)$$

for all $x_i \in U_i$, $i \in [r]$.

General o-minimal case

Theorem (C., Peterzil, Starchenko)

Let \mathcal{M} be an o-minimal expansion of \mathbb{R} . Assume $r \geq 3$, $R \subseteq X_1 \times \cdots \times X_r$ are definable with $\dim(X_i) = m$, and the projection of R to any $r - 1$ coordinates is finite-to-one. Then **exactly** one of the following holds.

1. For any finite $A_i \subseteq_n X_i$ **in general position**, $i \in [r]$, we have

$$|R \cap (A_1 \times \cdots \times A_r)| = O_R(n^{r-1-\gamma}),$$

for $\gamma = \frac{1}{8m-5}$ if $s \geq 4$, and $\gamma = \frac{1}{16m-10}$ if $s = 3$.

2. There exist definable relatively open sets $U_i \subseteq X_i$, $i \in [s]$, **an abelian Lie group** $(G, +)$ of dimension m and an open neighborhood $V \subseteq G$ of 0, and definable homeomorphisms $\pi_i : U_i \rightarrow V$, $i \in [s]$, such that for all $x_i \in U_i$, $i \in [s]$

$$\pi_1(x_1) + \cdots + \pi_s(x_s) = 0 \Leftrightarrow R(x_1, \dots, x_s).$$

Remarks

1. If \mathcal{M} is σ -minimal but is not elementarily equivalent to an expansion of \mathbb{R} — only get correspondence with a type-definable group.
2. One ingredient — improved Zarankiewicz bounds in distal structures discussed above (the power saving γ in the non-group case corresponds to the improvement on the basic Zarankiewicz bound).
3. Another — a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a “generic chunk”. We discuss a simple purely combinatorial case sufficient for σ -minimal:

Recognizing groups, 1

1. Assume that $(G, +, 0)$ is an abelian group, and consider the r -ary relation $R \subseteq \prod_{i \in [r]} G$ given by $x_1 + \dots + x_r = 0$.
2. Then R is easily seen to satisfy the following two properties, for any permutation of the variables of R :

$$\forall x_1, \dots, \forall x_{r-1} \exists! x_r R(x_1, \dots, x_r), \quad (\text{P1})$$

$$\forall x_1, x_2 \forall y_3, \dots, y_r \forall y'_3, \dots, y'_r \left(R(\bar{x}, \bar{y}) \wedge R(\bar{x}, \bar{y}') \rightarrow \right. \\ \left. (\forall x'_1, x'_2 R(\bar{x}', \bar{y}) \leftrightarrow R(\bar{x}', \bar{y}')) \right). \quad (\text{P2})$$

We show a converse, assuming $r \geq 4$:

Recognizing groups, 2

Theorem (C., Peterzil, Starchenko)

Assume $r \in \mathbb{N}_{\geq 4}$, X_1, \dots, X_r and $R \subseteq \prod_{i \in [r]} X_i$ are sets, so that R satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group $(G, +, 0_G)$ and bijections $\pi_i : X_i \rightarrow G$ such that for every $(a_1, \dots, a_r) \in \prod_{i \in [r]} X_i$ we have

$$R(a_1, \dots, a_r) \iff \pi_1(a_1) + \dots + \pi_r(a_r) = 0_G.$$

- ▶ If $X_1 = \dots = X_r$, property (P1) is equivalent to saying that the relation R is an $(r - 1)$ -dimensional permutation on the set X_1 , or a *Latin $(r - 1)$ -hypercube*, as studied by Linial and Luria. Thus the condition (P2) characterizes, for $r \geq 3$, those Latin r -hypercubes that are given by the relation " $x_1 + \dots + x_{r-1} = x_r$ " in an abelian group.
- ▶ If R is definable and X_i are type-definable in a (saturated) \mathcal{M} , then G is type-definable and π_i are relatively definable in \mathcal{M} .

Recognizing groups in the stable case

- ▶ In the stable case of the theorem, we only get “generic correspondence” with a type-definable group.
- ▶ An r -gon is a tuple a_1, \dots, a_r such that any $r - 1$ of its elements are (forking-)independent, and any element in it is in the algebraic closure of the other ones.
- ▶ An r -gon is *abelian* if, after any permutation of its elements, we have $a_1 a_2 \downarrow_{\text{acl}(a_1 a_2) \cap \text{acl}(a_3 \dots a_r)} a_3 \dots a_r$.
- ▶ If (G, \cdot) is a type-definable abelian group, g_1, \dots, g_{r-1} are independent generics in G and $g_r := g_1 \cdot \dots \cdot g_{r-1}$, then g_1, \dots, g_r is an abelian r -gon (associated to G).
- ▶ Conversely,

Theorem (C., Peterzil, Starchenko; independently Hrushovski)

Let $r \geq 4$ and a_1, \dots, a_r be an abelian r -gon. Then there is a type-definable (in \mathcal{M}^{eq}) connected abelian group (G, \cdot) and an abelian r -gon g_1, \dots, g_r associated to G , such that after a base change each g_i is interalgebraic with a_i .

Recognizing fields

- ▶ For the semialgebraic $K_{2,2}$ -free point-line incidence relation $R = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1 x_1 + y_2\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have the (optimal) lower bound $|R \cap (V_1 \times V_2)| = \Omega(n^{\frac{4}{3}})$.
- ▶ To define it we use both addition and multiplication, i.e. the field structure.
- ▶ This is not a coincidence — any non-trivial lower bound on the Zarankiewicz exponent of R allows to recover a field from it:

Theorem (Basit, C., Starchenko, Tao, Tran)

Assume that $\mathcal{M} = (M, <, \dots)$ is o-minimal and $R \subseteq M_{x_1} \times \dots \times M_{x_r}$ is a definable relation which is $K_{k, \dots, k}$ -free, but $|R \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})$. Then a real closed field is definable in the first-order structure $(M, <, R)$.

Ingredients

- ▶ An almost optimal Zarankiewicz bound for hypergraphs definable in locally modular \mathcal{o} -minimal expansions of groups, so e.g. for semilinear (= definable in $(\mathbb{R}, <, +)$) hypergraphs.
- ▶ The trichotomy theorem for \mathcal{o} -minimal structures [Peterzil, Starchenko'98].

Geometric weakly locally modular theories

- ▶ Recall that a complete first-order theory T is *geometric* if, in any model $\mathcal{M} \models T$, the algebraic closure operator satisfies the *Exchange Principle* and the quantifier \exists^∞ is eliminated.
- ▶ Hence, in a model of a geometric theory, acl defines a well-behaved notion of independence \perp .
- ▶ [Berenstein, Vassiliev] A geometric theory is (*weakly*) *locally modular* if for any small subsets $A, B \subseteq \mathbb{M} \models T$ there exists some small set $C \perp_\emptyset AB$ such that $A \perp_{\text{acl}(AC) \cap \text{acl}(BC)} B$.
- ▶ Intuition: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.
- ▶ E.g. any \mathcal{o} -minimal theory T is geometric, and T is weakly locally modular if and only if T is linear (i.e. any normal interpretable family of plane curves in T has dimension ≤ 1).

Zarankiewicz bound for semilinear relations

Theorem (Basit, C., Starchenko, Tao, Tran)

Let \mathcal{M} be an o-minimal locally modular expansion of a group and R a definable relation of arity $r \geq 2$, and $K_{k,\dots,k}$ -free. Then

$$|R \cap \prod_{i \in [r]} V_i| = O_{R,k,\varepsilon}(n^{r-1+\varepsilon})$$

for any $\varepsilon > 0$.

More precisely, there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = \beta(r, s) \in \mathbb{N}$, where s is the **description complexity** of R , such that

$$|R \cap \prod_{i \in [r]} V_i| \leq \alpha n^{r-1} (\log n)^\beta.$$

We can take $\beta(r, s) := s(2^{r-1} - 1)$.

Corollary for semilinear hypergraphs

Corollary

For every $s, k \in \mathbb{N}$ there exist some $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta(r, s) := s(2^{r-1} - 1)$ satisfying the following.

Suppose that $r \geq 2$, $d = d_1 + \dots + d_r \in \mathbb{N}$ and $R \subseteq \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_r}$ is semilinear and defined by $\leq s$ linear equalities and inequalities.

Then for every $K_{k, \dots, k}$ -free r -hypergraph $H \in \mathcal{F}_R$ we have

$$|E| \leq \alpha n^{r-1} (\log n)^\beta .$$

An application to incidences with polytopes

- ▶ Applying with $r = 2$ we get the following:

Corollary

For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.

Let $d \in \mathbb{N}$ and $H_1, \dots, H_q \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in \mathbb{R}^d . Let \mathcal{F} be the (infinite) family of all polytopes in \mathbb{R}^d cut out by arbitrary translates of H_1, \dots, H_q .

For any set V_1 of n_1 points in \mathbb{R}^d and any set V_2 of n_2 polytopes in \mathcal{F} , if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$ -free, then it contains at most $\alpha n (\log n)^q$ incidences.

- ▶ In particular (a similar result was obtained independently by [Tomon, Zakharov]):

Corollary

For any set V_1 of n_1 points and any set V_2 of n_2 (solid) boxes with axis parallel sides in \mathbb{R}^d , if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$ -free, then it contains at most $O_{d,k}(n(\log n)^{2d})$ incidences.

Dyadic rectangles and a lower bound

- ▶ Is the logarithmic factor necessary?
- ▶ We focus on the simplest case of incidences with rectangles with axis-parallel sides in \mathbb{R}^2 . The previous corollary gives the bound $O_{d,k}(n(\log n)^4)$.
- ▶ A box is *dyadic* if it is the direct products of intervals of the form $[s2^t, (s+1)2^t)$ for some integers s, t .
- ▶ Using a different argument, restricting to dyadic boxes we get a stronger upper bound $O\left(n \frac{\log n_1}{\log \log n_1}\right)$, and give a construction showing a matching lower bound (up to a constant).
- ▶ [Tomon, Zakharov] use our construction to disprove a conjecture of Alon, Basavaraju, Chandran, Mathew, and Rajendraprasad regarding the maximal possible number of edges in a graph of bounded separation dimension.

Problem

What is the optimal bound on the power of $\log n$? In particular, does it have to grow with the dimension d ?

Bound for $K_{k,\dots,k}$ -free relations in geometric weakly locally modular structures

- ▶ In our bounds, we can get rid of the logarithmic factor entirely restricting to the family of all finite r -hypergraphs induced by a given $K_{k,\dots,k}$ -free relation (as opposed to all $K_{k,\dots,k}$ -free r -hypergraphs induced by a given relation).

Theorem (Basit, C., Starchenko, Tao, Tran)

Assume that T is a geometric, weakly locally modular theory, and $\mathcal{M} \models T$. Assume that $r \in \mathbb{N}_{\geq 2}$ and $R \subseteq M_{x_1} \times \dots \times M_{x_r}$ is definable and $K_{k,\dots,k}$ -free. Then there exists some $\alpha = \alpha(R) > 0$ such that for every $H \in \mathcal{F}_R$ we have

$$|E| \leq \alpha n^{r-1}.$$

Moreover, if T is distal, then can relax “ $K_{k,\dots,k}$ -free” to “does not contain the direct product of r infinite sets”.

A related observation was made by Evans in the binary case for certain stable theories.

Recovering a field in the o-minimal case

Fact (Peterzil, Starchenko'98)

Let \mathcal{M} be an o-minimal saturated structure. TFAE:

- ▶ \mathcal{M} is not weakly locally modular;
- ▶ there exists a real closed field definable in \mathcal{M} .

- ▶ Combining this with the previous theorem, we thus get:

Corollary

Let \mathcal{M} be an o-minimal structure. TFAE:

- ▶ \mathcal{M} is weakly locally modular;
- ▶ For every definable $K_{k,\dots,k}$ -free R , every $H \in \mathcal{F}_R$ satisfies $|E| \leq \alpha n^{r-1}$.
- ▶ no infinite field is definable in \mathcal{M} .

Thank you!

- ▶ *Model-theoretic Elekes-Szabó for stable and o-minimal hypergraphs*, Artem Chernikov, Ya'acov Peterzil, Sergei Starchenko (arXiv:2104.02235)
- ▶ *Zarankiewicz's problem for semilinear hypergraphs*, Artem Chernikov, Abdul Basit, Sergei Starchenko, Terence Tao and Chieu-Minh Tran (arXiv:2009.02922)