Generalizations of stability and NTP_2

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Outline

Classification of first-order theories

Simple theories

NIP theories

NTP_2

Space of types

- Let T be a complete countable first-order theory, and we fix some very large saturated model $\mathbb{M}$ (a "universal domain").
- For a model $M \models T$, we let $\mathrm{Def}(M)$ be the Boolean algebra of definable subsets of M (with parameters).
- Let $S(M)$, the space of types over M, be the Stone dual of $\mathrm{Def}(M)$. I.e. the set of ultrafilters on $\mathrm{Def}(M)$ with the clopen basis consisting of sets of the form $[\phi] = \{p \in S(M) : \phi \in p\}$. It is a totally disconnected compact Hausdorff space.
- We abuse the notation slightly by not distinguishing between tuples of elements and singletons unless it matters.

General philosophy

- Shelah's philosophy of dividing lines: characterize complete first-order theories by their ability to encode certain combinatorial configurations.
- Analysis of definable sets (and types) vs analysis of models.
- Looking at algebraic structures such as groups or fields, the model-theoretic properties are usually closely related to algebraic properties.

Stable theories

Let $s_T(\kappa) = \sup\{|S(M)| : M \models T, |M| = \kappa\}$. Note that always $s_T(\kappa) \geq \kappa$.
T is called *stable* if any of the following equivalent properties hold:

- For every cardinal κ, $s_T(\kappa) \leq \kappa^{\aleph_0}$.
- There is some cardinal κ such that $s_T(\kappa) = \kappa$.
- There is no formula $\phi(x, y)$ and $(a_i)_{i \in \omega}$ (in some model) such that $\phi(a_i, a_j) \Leftrightarrow i < j$.

Examples

- Modules
- Algebraically closed fields
- Separably closed fields (C. Wood)
- Differentially closed fields
- Free groups (Z. Sela)
- Planar graphs (K. Podewski and M. Ziegler)

Dividing and Forking

Let $\phi(x, y)$ be a formula and A a set.

- We say that $\phi(x, a)$ *divides* over A if there is $k \in \omega$ and $(a_i)_{i \in \omega}$ such that $\text{tp}\,(a_i/A) = \text{tp}\,(a/A)$ and $\{\phi(x, a_i)\}_{i \in \omega}$ is k-inconsistent.
- Note that if $a \in A$ then $\phi(x, a)$ does not divide over A.
- We say that $\phi(x, a)$ *forks* over A if there are $\phi_0(x, a_0), \ldots, \phi_n(x, a_n)$ such that $\phi(x, a) \vdash \bigvee_{i \leq n} \phi_i(x, a_i)$ and $\phi_i(x, a_i)$ divides over A for each $i \leq n$.
- We say that a (partial) type $p(x)$ does not divide (fork) over A if it does not imply any formula which divides (forks) over A.

Note that the formulas forking over A form an ideal in $\text{Def}(\mathbb{M})$ generated by the formulas dividing over A.

Example

If μ is an A-invariant finitely additive probability measure on $\text{Def}\,(\mathbb{M})$ and $\mu(\phi(x, a)) > 0$ then $\phi(x, a)$ does not fork over A.

Forking in stable theories

Assume that T is stable.

1. Forking equals dividing: $\phi(x, a)$ forks over A if and only if it divides over A.
2. Let's write $a \mathop{\smile\!\!\!\!\!\!\!\;\mid\;}_c b$ when $\mathrm{tp}(a/bc)$ does not fork over c. Then $\mathop{\smile\!\!\!\!\!\!\!\;\mid\;}$ is a nice notion of independence (i.e. invariant under automorphisms of $\mathbb{M}$, symmetric, transitive, satisfies finite character, ...)
3. Assume that A is algebraically closed, in M^{eq}. Every $p \in S(A)$ has a unique non-forking extension $p' \in S(\mathbb{M})$ (i.e. $p \subseteq p'$ and that p' does not fork over A).

Use of forking

- Shelah's original purpose: to count the number of models a first-order theory may have. Essentially amounted to isolating the conditions for models to be classifiable by cardinal invariants.
- Geometric stability. Complexity of forking should be interrelated with the complexity of algebraic structures interpretable in the theory: trichotomy, group configuration, ...

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Simple theories

- A combinatorial definition: “not being able to encode a tree by some formula”.
- Equivalently, every $p \in S(\mathbb{M})$ does not fork over some countable subset $A \subset \mathbb{M}$.
- Introduced by Shelah for purely model-theoretic reasons trying to characterize existence of certain limit models.
- Later work of Hrushovski and Hrushovski-Cherlin in the special case rank 1.
- Kim and Pillay carried out the analysis in the general case.

Examples

- The theory of the random Rado graph.
- Pseudo-finite fields.
- ACFA (and in general stable theories with some random "noise").

Forking: Simple theories

1. Forking equals dividing: $\phi(x, a)$ forks over A if and only if it divides over A.
2. $\downarrow$ is still a nice notion of independence (symmetric, transitive, ...)
3. Stationarity and definability of types fail, types may have unboundedly many non-forking extensions.

(1) and (2) are due to Kim. Does anything of (3) survive?

Independence theorem

Turns out that the uniqueness of non-forking extensions can be replaced by an amalgamation statement.

Fact

Independence theorem over models (Hrushovski in the finite rank case, Kim and Pillay in full generality):
Assume that $a_1 \mathop{\smile\!\!\!\!\!\!|}_M b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!|}_M b_2$ *and* $tp(a_1/M) = tp(a_2/M)$.
Then there is $a \mathop{\smile\!\!\!\!\!\!|}_M b_1 b_2$ *and s.t.* $tp(ab_i/M) = tp(a_i b_i/M)$ *for* $i = 1, 2$.

In fact, existence of a relation satisfying (2) and the independence theorem implies that the theory is simple and that this relation is given by non-forking.

Key example: ACFA and geometric simplicity

1. Analysis of the theory ACFA by Chatzidakis, Hrushovski and Peterzil.
2. Independence is given by: $a \mathop{\smile\!\!\!\!\!\!|}_c b$ if and only if $\mathrm{acl}_\sigma(ac)$ is algebraically independent from $\mathrm{acl}_\sigma(bc)$ over $\mathrm{acl}_\sigma(c)$.
3. Trichotomy for sets of rank 1 holds.

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NIP

- A theory is NIP (No independence property) if it cannot "encode the random bipartite graph by a formula".
- NIP is equivalent to the finite Vapnik-Chervonenkis dimension of the families of φ-definable sets for all φ.
- We remark that if a theory is both simple and NIP, then it is stable.

Examples

- linear orders and trees
- ordered abelian groups (Gurevich-Schmitt)
- any o-minimal theory
- algebraically closed valued fields (and in fact any c-minimal theory)
- $\mathbb{Q}_p$

Forking in NIP

- Symmetry of $\downarrow$ fails badly – linear order.
- Some weaker replacements of stationarity:
  - A type $p \in S(\mathbb{M})$ does not fork over M if and only if it is invariant over M, i.e. $\varphi(x, a) \in p$ and $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$ implies $\varphi(x, b) \in p$. It follows that every type has boundedly many non-forking extensions.
  - Some forms of definability of types remain (uniform definability of types over finite sets, joint work with P. Simon).
- What about forking vs dividing? May fail over some sets.
- However, Pillay posed the problem whether forking equals dividing over models in NIP.

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Definition

We say that $\phi(x, y)$ has TP_2 if there are $(a_{i,j})_{i,j\in\omega}$ and $k \in \omega$ such that:

- $\{\phi(x, a_{i,j})\}_{j\in\omega}$ is k-inconsistent for every $i \in \omega$,
- $\{\phi(x, a_{i,f(i)})\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$.

T is called NTP_2 if no formula has TP_2.

- Every simple or NIP theory is NTP_2, but there is much more.
- To make sure that T is NTP_2 it is enough to check it for all formulas $\varphi(x, y)$ in which x is a singleton.

Example 1: Ultraproducts of p-adics

- Consider the valued field $\mathbf{K} = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathfrak{U}$, where $\mathfrak{U}$ is a non-principal ultrafilter.
- The theory of **K** is not simple: because the value group is linearly ordered.
- The theory of **K** is not NIP: the residue field is pseudo-finite, thus has the independence property by a result of J.L. Duret.
- Even in the pure field language, as the valuation ring is definable uniformly in p (J. Ax).

Ax-Kochen for NTP_2

However, $\mathbf{K}$ is NTP_2 by the following:

Theorem

Let $\mathbf{K} = (K, k, \Gamma)$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that k is NTP_2. Then $\mathbf{K}$ is NTP_2.

Analogous to the theorem of F. Delon for NIP.

Example 2: Valued difference fields

- We consider valued difference fields $\mathbf{K} = (K, k, \Gamma, \sigma)$ of equicharacteristic 0.
- Kikyo-Shelah: It T has the Strict Order Property (which is the case with valued fields), then the model companion of $T \cup \{\sigma \text{ is an automorphism}\}$ does not exist.
- However, if we impose in addition that σ is contractive (i.e. $v(\sigma(x)) > n \cdot v(x)$ for all $n \in \omega$), then the model companion VFA_0 exists. It is axiomatized by saying that (k, σ) is a model of ACFA_0, (Γ, σ) is a divisible $\mathbb{Z}[\sigma]$ module and $\mathbf{K}$ is σ-henselian.
- A natural model of VFA_0: non-standard Frobenius acting on an algebraically closed valued field of char 0.
- Again neither simple nor NIP.

Example 2: Valued difference fields

Theorem

(Ch., M. Hils) Let $\mathbf{K} = (K, k, \Gamma, \sigma)$ *be a* σ*-henselian contractive valued difference field of equicharacteristic* 0*. Assume that both* (k, σ) *and* (Γ, σ) *are* NTP_2*. Then* $\mathbf{K}$ *is* NTP_2*.*

The proof utilizes the analysis of S. Azgin and properties of indiscernible arrays to reduce the situation to the previous example.

Forking in NTP2

- Back to Pillay's question: is forking = dividing over models in NIP theories?
- NTP_2 turned out to be the right context for clarifying this.
- We say that a set A is an *extension base* if every $p \in S(A)$ does not fork over A. E.g. every model is an extension base, in any theory. In simple theories, o-minimal theories or c-minimal theories, every set is an extension base.

Theorem

(Ch., I. Kaplan) Let A be an extension base in an NTP_2 theory T. Then $\phi(x, a)$ divides over A if and only if it forks over A.

Forking in NTP2

- The reason: existence of strictly invariant types.
- A type $p(x) \in S(\mathbb{M})$ is called *strictly invariant* over A if it is invariant (i.e. $\phi(x, a) \in p$ and $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ implies $\phi(x, b) \in p$) and for every small $A \subseteq B \subseteq \mathbb{M}$, if $c \models p|_B$ then $\mathrm{tp}(B/cA)$ does not fork over A.
- E.g. every generically stable type or every invariant type in a simple theory are strictly invariant.
- The crucial step of the proof is to show that in NTP_2 theories every type $p(x)$ over a model M has a global strictly invariant extension $q(x)$ (the so called Broom lemma).
- Then one can show that if $\varphi(x, a)$ divides over M, $p(x) \in S(\mathbb{M})$ is a strictly invariant extension and $(a_i)_{i \in \omega}$ is a Morley sequence in q (i.e. $a_i \models q|_{a_{<i}M}$) then $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent.

Weak independence theorem

- Recall the amalgamation of types in simple theories.
- Of course, fails in the presence of a linear order.
- In his work on approximate subgroups, Hrushovski found a reformulation of the independence theorem which makes sense in the context where $\mathop{\smile\!\!\!\!\!\!\mid}$ is not symmetric.
- Combining it with some new results on forking in NTP_2 (specifically that the forking ideal is S1) we get:

Weak independence theorem

Theorem

(I. Ben Yaacov, Ch.) Let T be NTP_2 and A an extension base. Assume that $c \mathop{\smile\!\!\!\!\!\!|}_M ab$, $a \mathop{\smile\!\!\!\!\!\!|}_M bb'$ and $b \equiv_M b'$. Then there is c' such that $c' \mathop{\smile\!\!\!\!\!\!|}_M ab'$, $c'a \equiv_M ca$, $c'b' \equiv_M cb$.

Remains valid over extension bases, but with Lascar-strong type in the place of type. In fact, can be used to deduce that Lascar strong type equals Kim-Pillay strong type over extension bases in NTP_2 theories. Gives rise to some results on stabilizers.

Summary

So why should one care about NTP_2?

- Empirical argument: every dividing line for first-order theories introduced by Shelah eventually becomes important.
- Methodical argument: allows for uniform proofs of results in simple and NIP theories, but also arises naturally trying to understand some special cases.
- Forking works.
- Important examples.