Idempotent Keisler measures

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Logic Seminar at Imperial College, London, UK (via Zoom)
May 19, 2021
Spaces of types

- Let $T$ be a complete first-order theory in a language $\mathcal{L}$, $\mathbb{M} \models T$ a monster model (i.e. $\kappa$-saturated and $\kappa$-homogeneous for a sufficiently large cardinal $\kappa$), $\mathcal{M} \preceq \mathbb{M}$ a small elementary submodel.

- For $A \subseteq \mathbb{M}$ and $x$ an arbitrary tuple of variables, $S_x(A)$ denotes the set of complete types over $A$.

- Let $\mathcal{L}_x(A)$ denote the set of all formulas $\varphi(x)$ with parameters in $A$, up to logical equivalence — which we identify with the Boolean algebra of $A$-definable subsets of $\mathbb{M}_x$; $\mathcal{L}_x := \mathcal{L}_x(\emptyset)$.

- Then the types in $S_x(A)$ are the ultrafilter on $\mathcal{L}_x(A)$.

- By Stone duality, $S_x(A)$ is a totally disconnected compact Hausdorff topological space with a basis of clopen sets of the form

  $$\langle \varphi \rangle := \{ p \in S_x(A) : \varphi(x) \in p \}$$

  for $\varphi(x) \in \mathcal{L}_x(A)$.

- We refer to types in $S_x(\mathbb{M})$ as global types.
Keisler measures

- A *Keisler measure* $\mu$ in variables $x$ over $A \subseteq M$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of $A$-definable subsets of $M_x$.
- $\mathcal{M}_x(A)$ denotes the set of all Keisler measures in $x$ over $A$.
- Then $\mathcal{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n}\{\mu \in \mathcal{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A)$, $r_i, s_i \in [0,1]$ for $i < n$.

- Identifying $p$ with the Dirac measure $\delta_p$, $S_x(A)$ is a closed subset of $\mathcal{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathcal{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int fd\mu_i \to \int fd\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$. 
Some examples of Keisler measures

1. In arbitrary $T$, given $p_i \in S_x(A)$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ with $\sum_{i \in \mathbb{N}} r_i = 1$, $\mu := \sum_{i \in \mathbb{N}} r_i \delta_{p_i} \in M_x(A)$.

2. Let $T = \text{Th}(\mathbb{N}, =)$, $|x| = 1$. Then

$$S_x(M) = \{ \text{tp}(a/M) : a \in M \} \cup \{ p_\infty \},$$

where $p_\infty$ is the unique non-realized type axiomatized by $\{ x \neq a : a \in M \}$. By QE, every formula is a Boolean combination of $\{ x = a : a \in M \}$, from which it follows that every $\mu \in M_x(M)$ is as in (1).

3. More generally, if $T$ is $\omega$-stable (e.g. strongly minimal, say $\text{ACF}_p$ for $p$ prime or 0) and $x$ is finite, then every $\mu \in M_x(M)$ is a sum of types as in (1).

4. Let $T = \text{Th}(\mathbb{R}, <)$, $\lambda$ be the Lebesgue measure on $\mathbb{R}$ and $|x| = 1$. For $\varphi(x) \in L_x(M)$, define $\mu(\varphi) := \lambda(\varphi(M) \cap [0, 1]_{\mathbb{R}})$ (this set is Borel by QE). Then $\mu(X)$ is a Keisler measure, but not a sum of types as in (1).
Brief history of the theory of Keisler measures

- Measures and forking in stable/NIP theories [Keisler’87]
- Automorphism-invariant measures in $\omega$-categorical structures [Albert’92, Ensley’96]
- Applications to neural networks [Karpinski, Macintyre’00]
- Pillay’s conjecture and compact domination [Hrushovski, Peterzil, Pillay’08], [Hrushovski, Pillay’11], [Hrushovski, Pillay, Simon’13]
- Randomizations [Ben Yaacov, Keisler’09] (NIP and stability are preserved)
- Approximate Subgroups [Hrushovski’12]
- Definably amenable NIP groups [C., Simon’15] (in particular translation-invariant measures are classified)
- Tame (equivariant) regularity lemmas: subsets of [C., Conant, Malliaris, Pillay, Shelah, Starchenko, Terry, Tao, Towsner, …’11–…]

Keisler measures outside of NIP?

- All of the above — mostly inside the context of NIP theories (thanks to the VC-theory, measures are strongly approximated by types).
- Pseudofinite fields — ultraproducts of finite counting measures are very well-behaved (more generally in MS-measurable structures).
- But very few general results outside of NIP so far. Some counterexamples:
  - Independent product $\otimes$ of Borel-definable measures is not associative in general [Conant, Gannon, Hanson’21];
  - Not all groups in simple theories are definably amenable [C., Hrushovski, Kruckman, Krupinski, Moconja, Pillay and Ramsey’21].
- Some positive results:
  - A weak generalization of $\varepsilon$-nets for $n$-dependen theories [C., Towsner’20]
  - NSOP$_1$ is preserved under randomizations [Ben Yaacov, C., Ramsey, 21+]
Independent product of definable types $\otimes$, 1

- Given two global types $p(x), q(y)$, there are usually many different global types $r(x, y)$ satisfying $r(x, y) \supseteq p(x) \cup q(y)$ (as $L_x(\mathbb{M}) \times L_y(\mathbb{M}) \subseteq L_{xy}(\mathbb{M})$).

- Under additional assumptions on $p$, there is often a canonical “generic” choice of $r$ not introducing any dependencies between $x$ and $y$ (e.g. not containing $x = y$). We restrict to definable types for simplicity of presentation (but works for invariant types as well).

- Given $A \subseteq B \subseteq \mathbb{M}$, a type $p \in S_x(B)$ is definable over $A$ if for every formula $\varphi(x, y) \in L_{xy}$ there exists a formula $d_p \varphi(y) \in L_y(A)$ such that

$$\forall b \in B^y, \varphi(x, b) \in p \iff \models d_p \varphi(b).$$

- A global type is definable if it is definable over some small model.

- A theory is stable if and only if all types are definable [Shelah].
Independent product of definable types $\otimes$, 2

- Assume that $p \in S_x(\mathbb{M})$, $q \in S_y(\mathbb{M})$ and $p$ is definable. Then $p \otimes q \in S_{xy}(\mathbb{M})$ is defined via

$$\varphi(x, y) \in p \otimes q \iff d_p \varphi(y) \in q$$

for every $\varphi(x, y) \in \mathcal{L}_{xy}$.

- Equivalently, $p \otimes q = tp(a, b/\mathbb{M})$ for some/any $b \models q$ and $a \models p'|_{\mathbb{M}b}$ (in some $\mathbb{M}' \succ \mathbb{M}$; where $p' \in S_x(\mathbb{M}')$ is the extension of $p$ given by the same definition schema).

- E.g. if $p$ is the non-realized type in $\text{Th}(\mathbb{N}, =)$, then $p(x) \otimes p(y) = p(y) \otimes p(x)$ is axiomatized by

$$\{x \neq a, y \neq a : a \in \mathbb{M}\} \cup \{x \neq y\}.$$

- Assume $p(x) = \{x > a : a \in \mathbb{M}\}$ in $\text{Th}(\mathbb{Q}, <)$. Then

$$p(x) \otimes q(y) = \{x > a, y > a : a \in \mathbb{M}\} \cup \{x > y\} \neq p(y) \otimes q(x).$$

- Hence $\otimes$ is associative, but not commutative (unless $T$ is stable).
Convolution product $\ast$ of definable types

- Assume now that $T$ expands a group, i.e. there exists a definable functions $\cdot$ such that for some/any $\mathcal{M} \models T$, $(\mathcal{M}_x, \cdot)$ is a group.

- In this case, given definable $p, q \in S_x(\mathbb{M})$, we have a definable type $p \ast q \in S_x(\mathbb{M})$ via

$$\varphi(x) \in p \ast q \iff \varphi(x \cdot y) \in p(x) \otimes q(y)$$

for every $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$.

- Equivalently, $p \ast q = \text{tp}(a \cdot b/\mathbb{M})$ for some/any $(a, b) \models p \otimes q$ in a larger monster model.

- Let $S_x^{\text{def}}(\mathbb{M})$ be the set of all definable global types. Then $(S_x^{\text{def}}(\mathbb{M}), \ast)$ is a left-continuous semigroup.

- “Left continuous” means: the map $- \ast q : S_x^{\text{def}}(\mathbb{M}) \to S_x^{\text{def}}(\mathbb{M})$ is continuous for every fixed $q \in S_x^{\text{def}}(\mathbb{M})$. 
Idempotent types

- A type \( p \in S_x^{\text{def}}(M) \) is idempotent if \( p \ast p = p \).
- E.g. let \( M \) be \((\mathbb{Z}, +, P_{n,\alpha})\), with \((P_{n,\alpha} : \alpha < 2^{\aleph_0})\) naming all subsets of \( \mathbb{Z}^n \), for all \( n \).
  Then all types over \( M \) are trivially definable, and idempotent types are precisely the idempotent ultrafilters in the sense of Galvin–Glazer’s proof of Hindman’s theorem (for every finite partition of \( \mathbb{Z} \), some part contains all finite sums of elements of an infinite set), see e.g. [Andrews, Goldbring’18].
- In stable theories, idempotent types are known to arise from type-definable subgroups (group chunk theorem and its variants [Hrushovski, Newelski]).
- This is parallel to the following classical line of research:
Motivation: analogy with the classical (locally-)compact case

- Let $G$ be a locally compact topological group.
- Then the space of regular Borel probability measures on $G$ is equipped with the convolution product:

\[
\mu \ast \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y) d\mu(x) d\nu(y)
\]

for a Borel set $A \subseteq G$.

- If $G$ is compact, then $\mu$ is idempotent if and only if the support of $\mu$ is a compact subgroup of $G$ and $\mu$ restricted to it is the (bi-invariant) Haar measure [Wendel’54].
- Same characterization extends to locally compact abelian groups [Rudin’59, Cohen’60].
- Compact (semi-)topological semigroup — the picture becomes more complicated [Glicksber’59, Pym’69, ...].
Independent product \( \otimes \) of definable Keisler measures

- We would like to find a parallel for Keisler measures, generalizing the situation for types. First, need to make sense of the convolution product.
- A Keisler measure \( \mu \in \mathcal{M}_x(\mathcal{M}) \) is **definable** (over \( \mathcal{M} \preceq \mathcal{M} \)) if:
  1. for any \( \varphi(x, y) \in \mathcal{L}_{xy} \) and \( b \in \mathcal{M}_y \), \( \mu(\varphi(x, b)) \) depends only on \( \text{tp}(b/\mathcal{M}) \)
     (in which case, given \( q \in S_y(\mathcal{M}) \), we write \( \mu(\varphi(x, q)) \) to denote \( \mu(\varphi(x, b)) \) for some/any \( b \models q \));
  2. the map \( q \in S_y(\mathcal{M}) \mapsto \mu(\varphi(x, q)) \in [0, 1] \) is continuous.
- A type \( p \in S_x(\mathcal{M}) \) is definable as a type iff it is definable as a measure.
- Given \( \mu \in \mathcal{M}_x(\mathcal{M}), \nu \in \mathcal{M}_y(\mathcal{M}) \) with \( \mu \mathcal{M}\)-definable, we can define \( \mu \otimes \nu \in \mathcal{M}_{xy}(\mathcal{M}) \) via
  \[
  \mu \otimes \nu(\varphi(x, y)) := \int_{S_y(\mathcal{M})} \mu(\varphi(x, q)) d\nu|_\mathcal{M}(q).
  \]
- The integral makes sense by (2), viewing \( \nu|_\mathcal{M} \) as a regular Borel measure on \( S_y(\mathcal{M}) \). (Works also for only *Borel-definable*).
Convolution product $\ast$ of definable Keisler measures

- $\otimes$ on definable measures extends $\otimes$ on definable types defined earlier.

- If now $T$ expands a group, given definable $\mu, \nu \in \mathcal{M}_x(\mathbb{M})$, we get a definable $\mu \ast \nu \in \mathcal{M}_x(\mathbb{M})$ via

$$\mu \ast \nu(\varphi(x)) := \mu_x \otimes \nu_y(\varphi(x \cdot y)).$$

- Again, restricting to definable types, we recover $\ast$ defined earlier.

- The set of all definable Keisler measures with $\ast$ is a semigroup. A measure $\mu$ is idempotent if $\mu \ast \mu = \mu$.

**Theorem (C., Gannon’20)**

*If $T$ is NIP, then $\ast$ is again left-continuous (on invariant measures).*

- In general $T$ — unclear.
Idempotent Keisler measures vs the classical locally compact case

- First of all, in general a definable group has no non-discrete topology.
- Given $\mu \in \mathcal{M}_x(A)$, its support is

$$S(\mu) := \{ p \in S_x(A) : \varphi(x) \in p \implies \mu(\varphi(x)) > 0 \}.$$  

It is a closed non-empty subset of $S_x(A)$.
- As we mentioned, in a locally compact topological group, support of an idempotent measure is a closed subgroup — no longer true for idempotent Keisler measures (with respect to $\ast$ on types), even if there is some nice topology present:
Supports of idempotent Keisler measures: an example, 1

- Let $\mathcal{M} = (S^1, \cdot, C(x, y, z))$ be the compact unit circle group (of rotations) over $\mathbb{R}$, with $C$ the cyclic clockwise ordering.
- Let $\mu \in \mathcal{M}_x(\mathbb{M})$ be given by $\mu(\varphi(x)) = h(\varphi(\mathcal{M}))$ for $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, where $h$ is the Haar measure on $S^1$.
- Then $\mu$ is definable and right translation invariant (by elements of $\mathbb{M}$), hence idempotent.
- Let $\text{st} : S_x(\mathbb{M}) \to \mathcal{M}$ be the standard part map. Assume that $p \in S(\mu)$ and $\text{st}(p) = a$. Then $\varphi_\varepsilon(x) := C(a - \varepsilon, x, a + \varepsilon) \notin p$ for every infinitesimal $\varepsilon \in \mathbb{M}$ ($x \neq a \in p$ as $h(x = a) = 0$, and if $\varphi_\varepsilon(x) \in p$, then $\mu(\varphi_\varepsilon(x) \land x \neq a) > 0$, but $\varphi_\varepsilon(\mathcal{M}) = \{a\}$ — a contradiction).
- As the types in $S_x(\mathbb{M})$ are determined by the cuts in the circular order, it follows that for every $a \in \mathcal{M}$ there are exactly two types $a_+(x), a_-(x) \in S(\mu)$ determined by whether $C(a + \varepsilon, x, b)$ holds for every infinitesimal $\varepsilon$ and $b \in \mathcal{M}$, or $C(b, x, a - \varepsilon)$ holds for every infinitesimal $\varepsilon$ and $b \in \mathcal{M}$, respectively.
It follows that \((S(\mu), \ast) \cong S^1 \times \{+, -\}\) with multiplication defined by:
\[ a_\delta \ast b_\gamma = (a \cdot b)_\delta \]
for all \(a, b \in S^1\) and \(\delta, \gamma \in \{+, -\}\).

Hence \((S(\mu), \ast)\) is not a group (as it has two idempotents).

This group is NIP (definable in an \(o\)-minimal theory), unstable.
Supports of idempotent Keisler measures: a theorem

- Adapting Glicksberg, we show:

**Theorem (C., Gannon’20)**

1. *(T arbitrary)* Let \( \mu \in \mathcal{M}_x(\mathcal{M}) \) be an idempotent definable and invariantly supported Keisler measure. Then \((S(\mu), \ast)\) is a compact, left continuous semigroup with no closed two-sided ideals.

2. *(T NIP)* The same conclusion holds just assuming that \( \mu \) is definable.

- Where:
  - \( I \subseteq S(\mu) \) is a left (right) ideal if: \( q \in I \implies p \ast q \in I \) (resp., \( q \ast p \in I \)) for every \( p \in S(\mu) \). Two-sided = both left and right.
  - \( \mu \) is **invariantly supported** if there exists a small model \( \mathcal{M} \preceq \mathcal{M} \) s.t. every \( p \in S(\mu) \) is \( \text{Aut}(\mathcal{M}/\mathcal{M}) \)-invariant.
Type-definable subgroups

- Instead of closed subgroups in the topological setting, we consider *type-definable* subgroups.
- Assume that $\mathbb{M} \models T$ expands a group, and $\mathcal{H}$ is a type-definable subgroup of $(\mathbb{M}, \cdot)$ (i.e. the underlying set of $\mathcal{H}$ can be defined by a small partial type $H(x)$ with parameters in $\mathbb{M}$).
- Let $\mathcal{H}$ be type-definable and suppose that $\mu \in \mathcal{M}_x(\mathbb{M})$ is concentrated on $\mathcal{H}$ (i.e. $p \in S(\mu) \implies p(x) \vdash H(x)$) and is right $\mathcal{H}$-invariant (i.e. for any $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, $a \in \mathcal{H}$, $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$). Then $\mu$ is idempotent.
- Ideology: by analogy with the classical case, we expect all idempotent Keisler measures in model-theoretically tame groups to be of this form.
- (Translation-invariant Keisler measures in NIP groups are classified: the ergodic ones are described as certain liftings of the Haar measure on the canonical compact quotient $G/G^{00}$ [C., Simon’18].)
Idempotent measures in stable groups

- Can confirm for stable groups:

**Theorem (C., Gannon’20)**

Let $T$ be a stable theory expanding a group and $\mu \in M_x(M)$ a Keisler measure. TFAE:

1. $\mu$ is idempotent;
2. $\mu$ is the unique right/left-invariant measure on its stabilizer, i.e. the type-definable subgroup $St(\mu) = \{ g \in M : g \cdot \mu = \mu \}$.

- The following groups are stable: abelian, free, algebraic over $\mathbb{C}$ (e.g. $GL_n(\mathbb{C}), SL_n(\mathbb{C})$, abelian varieties).

- Ingredients: structure of the supports of definable idempotent measures in NIP; definability of all measures in stable theories (and type-definability of their stabilizers); a variant of Hrushovski’s group chunk theorem for partial types due to Newelski.
Idempotent measures in NIP

► Can we classify idempotent measures in NIP, or even more generally?
► Conjecture: in a (definably amenable) NIP group, every idempotent definable (invariant) measure $\mu$ is a left-invariant measure on its type-definable (invariant) stabilizer subgroup.
► Note: no longer needs to be unique!
► Work in progress: can confirm under some additional assumptions: abelian group, $\mu$ generically stable (in which case it is the unique measure on its type-definable stabilizer).