Ergodic measures and genericity in definably amenable NIP groups

Artem Chernikov

(IMJ-PRG)

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Definable groups

- Let G be a definable group (i.e. a definable set with a definable group operation in some first-order structure M in some language L).
- G is equipped with a Boolean algebra of $L(M)$-definable subsets $\mathrm{Def}_G(M)$.
- Let the space of G-types $S_G(M)$ be the (compact, Hausdorff, totally disconnected) Stone dual of $\mathrm{Def}_G(M)$ (i.e. elements of $S_G(M)$ are ultrafilters on $\mathrm{Def}_G(M)$).
- $G(M)$ acts on $S_G(M)$ by homeomorphisms, a point transitive flow.
- Let $\mathbb{M} \succ M$ be a saturated "monster" model, let $G(\mathbb{M})$ be the interpretation of G in $\mathbb{M}$.

NIP and VC dimension

- NIP was introduced by Shelah for the purposes of his classification theory (motivated by questions like: given a theory T and uncountable κ, how many models of cardinality κ can it have?).
- Turned out to be closely connected to Vapnik–Chervonenkis dimension, or VC-dimension — a notion from combinatorics introduced around the same time (central in computational learning theory).

NIP and VC dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- The *VC dimension* of $\mathcal{F}$ is the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- An L-structure M is NIP if for every formula $\phi(x, y) \in L$, where x and y are tuples of variables, the family of definable subsets of M given by $\{\phi(x, a) : a \in M\}$ is of finite VC dimension (note that this is a property of T).
- This is a talk about groups definable in NIP structures.

Examples of NIP groups

- Any *o*-minimal structure is NIP, so e.g. groups definable in $(\mathbb{R}, +, \times)$ such as $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$, etc.
- Any stable structure is NIP, so e.g. algebraic groups over alrgebraically closed fields, but also free groups (in the pure group language) [Sela].
- $(\mathbb{Q}_p, +, \times, 0, 1)$ is NIP.
- Algebraically closed valued fields are NIP.

NIP groups and tame/null dynamical systems

- Turns out that the topological dynamics hierarchy is closely connected to the model theoretic hierarchy (independently noticed and explored by Ibarlucía).
- If G is an NIP group, then $G \curvearrowright S_G(M)$ is null (in the sense of Glasner-Megrelishvili).
- If G is a stable group, then $G \curvearrowright S_G(M)$ is WAP.
- Some of our results hold just assuming that $G \curvearrowright S_G(M)$ is tame, yet to be clarified (by compactness null = tame in this setting).

Connected components

- Working in $\mathbb{M}$, H is a *type-definable* subgroup of G if H is given by an intersection of a small family of definable sets (small means smaller than the saturation of $\mathbb{M}$).
- A type-definable group in general is not an intersection of definable groups (though true in stable groups).
- For a small set $A \subset \mathbb{M}$, $G_A^{00} = \bigcap \{H \leq G : H \text{ is type-definable over } A, \text{ of bounded index}\}$.
- [Shelah] Let G be an NIP group. Then $G_A^{00} = G_\emptyset^{00}$ for any small set $A \subseteq \mathbb{M}$.
- G^{00} is a normal type-definable subgroup of bounded index.

Logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map, we endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.
- With this topology, G/G^{00} is a compact topological group.

Example

1. If is a stable group, then G/G^{00} is a profinite group: it is the inverse image of the groups G/H, where H ranges over all definable subgroups of finite index.
   E.g. If $G = (Z, +)$, then G^{00} is the set of elements divisible by all n. The quotient G/G^{00} is isomorphic as a topological group to $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$.
2. If $G = \mathrm{SO}(2, \mathcal{R})$ is the circle group defined in a (saturated) real closed field $\mathcal{R}$, then G^{00} is the set of infinitesimal elements of G and G/G^{00} is isomorphic to the standard circle group $SO(2, \mathbb{R})$.

Keisler measures and definable amenability

- A *Keisler measure* μ is a finitely additive probability measure on the Boolean algebra $\mathrm{Def}_G(M)$.
- Every Keisler measure extends uniquely to a regular Borel probability measure on $S_G(M)$.
- A definable group G is *definably amenable* if it admits a G-invariant Keisler measure on $\mathrm{Def}_G(M)$.
- Note: this is a property of the definable group G, i.e. does not depend on M.

Examples of definably amenable groups

- Stable groups (in particular the free group F_2, viewed as a structure in a pure group language, is definably amenable).
- Definable compact groups in *o*-minimal theories or in *p*-adics (compact Lie groups, e.g. $SO(3, \mathbb{R})$, seen as definable groups in $\mathbb{R}$).
- Solvable NIP groups, or more generally any NIP group G such that $G(M)$ is amenable as a discrete group.
- $SL(n, \mathbb{R})$ is *not* definably amenable for $n > 1$.

Dynamics of $G \curvearrowright S_G(\mathbb{M})$: stable example

- Consider $G \curvearrowright S_G(\mathbb{M})$ for G a stable group.
- Then there is a unique minimal flow and it is homeomorphic to G/G^0. Moreover, the system is uniquely ergodic.
- The elements of the minimal flow are precisely the *generic* types.
- A set $X \in \mathrm{Def}_G(\mathbb{M})$ is *generic* (syndetic) if $G = \bigcup_{i \le n} g_i X$ for some $g_0, \ldots, g_n \in G$. A type $p \in S_G(\mathbb{M})$ is generic if every formula in it is generic.
- What about NIP? Consider $(\mathbb{R}, +)$. Any generic set must be unbounded on both sides, but then non-generic sets don't form an ideal and there are no generic types.
- Several alternative notions of genericity were suggested. Turns out that they all are equivalent in definably amenable NIP groups.

First option: weak generics

- [Newelski] A set $X \in \mathrm{Def}_G(\mathbb{M})$ is *weakly generic* if there is a non-generic $Y \in \mathrm{Def}_G(\mathbb{M})$ such that $X \cup Y$ is generic.
- A type $p \in S_G(\mathbb{M})$ is weakly generic if for every $\phi(x) \in p$, the set $\phi(\mathbb{M})$ is weakly generic.
- Weakly generic subsets of G always form a filter in $\mathrm{Def}_G(\mathbb{M})$, so weakly generic types always exist.
- In fact, the set of weakly generic types is precisely the mincenter of $S_G(\mathbb{M})$, i.e. the closure of the union of all minimal flows.

Second option: f-generics

- By analogy with f-generics developed for groups in simple theories ("f" is for "forking").
- $X \in \mathrm{Def}_G(\mathbb{M})$ *divides* over M if there are $\sigma_i \in \mathrm{Aut}(\mathbb{M}/M)$ for $i \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\sigma_{i_1}(X) \cap \ldots \cap \sigma_{i_k}(X) = \emptyset$ for any $i_1 < \ldots < i_k$.
- [C., Kaplan] Assuming NIP, the set of all X dividing over M is an ideal in $\mathrm{Def}_G(\mathbb{M})$.
- We say that $X \in \mathrm{Def}_G(\mathbb{M})$ is f-generic if there is some small model M such that $g \cdot X$ does not divide over M for all $g \in G(\mathbb{M})$.
- A type $p \in S_G(\mathbb{M})$ is f-generic, if for every $\phi(x) \in p$, the set $\phi(\mathbb{M})$ is f-generic.

Characterization of definable amenability

Theorem

[C., Simon] Let G be an NIP group. The following are equivalent:

1. *G is definably amenable.*
2. *The family of non-f-generic sets is an ideal in $\mathrm{Def}_G(\mathbb{M})$.*
3. *There is an f-generic type $p \in S_G(\mathbb{M})$.*
4. *$G \curvearrowright S_G(\mathbb{M})$ has a bounded orbit (equivalently, the action of G on the space of measures on $S_G(\mathbb{M})$ has a bounded orbit).*

Generics in definably amenable NIP groups

Theorem

[C., Simon] Let G be a definably amenable NIP group.

1. *Let $X \in \mathrm{Def}_G(\mathbb{M})$, the following are equivalent:*
   - 1.1 *X is f-generic,*
   - 1.2 *X is weakly generic,*
   - 1.3 *$\mu(X) > 0$ for some G-invariant Keisler measure μ on $\mathrm{Def}_G(\mathbb{M})$,*
   - 1.4 *There is no infinite sequence (g_i) from G and $k \in \mathbb{N}$ such that $g_{i_1}X \cap \ldots \cap g_{i_k}X = \emptyset$ for all $i_1 < \ldots < i_k$.*
2. *Moreover, for $p \in S_G(\mathbb{M})$, the following are equivalent:*
   - 2.1 *p is f-generic,*
   - 2.2 $\mathrm{Stab}(p) = G^{00}$.
3. *G is uniquely ergodic if and only if it admits a generic type, in which case all notions above coincide with genericity.*

Finding measures from generic types

- Let $p \in S_G(\mathbb{M})$ be f-generic, and let h_0 be the (normalized) Haar measure on G/G^{00}.
- Let $p \in S_G(\mathbb{M})$ be f-generic (so in particular gp is G^{00}-invariant for all $g \in G$).
- Given $\phi(\mathbb{M}) \in \mathrm{Def}_G(\mathbb{M})$, let $A_{\phi,p} = \{\bar{g} \in G/G^{00} : \phi(x) \in g \cdot p\}$. It is a measurable subset of G/G^{00} (using Borel-definability of invariant types in NIP).
- For $\phi(x) \in L(\mathbb{M})$, we define $\mu_p(\phi(x)) = h_0(A_{\phi,p})$.
- Then μ_p is G-invariant Keisler measure on $\mathrm{Def}_G(\mathbb{M})$ (this generalizes a construction of Pillay and Hrushovski for p strongly f-generic).
- Note that $\mu_{g \cdot p} = \mu_p$ for any $g \in G$.
- We would like to understand the map $p \mapsto \mu_p$ better.

VC theorem

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a countable family of subsets of X of finite VC-dimension such that every $S \in \mathcal{F}$ is measurable. Then for every $\varepsilon > 0$ there is some $n = n(\varepsilon, \text{VC-dim}(\mathcal{F})) \in \mathbb{N}$ and some $x_1, \ldots, x_n \in X$ such that for any $S \in \mathcal{F}$ we have $\left|\mu(S) - \frac{|\{i : x_i \in S\}|}{n}\right| < \varepsilon$.

- Countability of $\mathcal{F}$ may be relaxed to the measurability of the maps
  - $(x_1, \ldots, x_n) \mapsto \sup_{S \in \mathcal{F}} \left|\mu(S) - \frac{|\{i : x_i \in S\}|}{n}\right|$ and
  - $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \sup_{S \in \mathcal{F}} \left|\frac{|\{i : x_i \in S\}|}{n} - \frac{|\{i : y_i \in S\}|}{n}\right|$.

"Equivariant" VC-theorem

- It follows that Keisler measures in NIP theories can be approximated by the averages of types:
- **Fact.** For any measure μ, formula $\phi(x, y) \in L$ and $\varepsilon > 0$ there are some $p_1, \ldots, p_n \in S(\mathbb{M})$ in the support of μ such that $\mu(\phi(x, a)) \approx^{\varepsilon} \frac{|\{i : \phi(x,a) \in p_i\}|}{n}$ for any $a \in \mathbb{M}$.
- We obtain some "equivariant" versions of the VC-theorem with respect to μ_p's, e.g.
- **Proposition.** Let μ be a G-invariant measure on $\mathrm{Def}_G(\mathbb{M})$. Then for every $\phi(x, y) \in L$ and $\varepsilon > 0$ there are some f-generic $p_1, \ldots, p_n \in S_G(\mathbb{M})$ such that $\mu(\phi(x, a)) \approx^{\varepsilon} \frac{\sum \mu_{p_i}(\phi(x,a))}{n}$ for any $a \in \mathbb{M}$.
- Our proof is by using the VC theorem with respect to the Haar measure on G/G^{00}. We work with an uncountable family of sets, so have to invoke universal measurability of analytic sets in Polish groups to ensure that the assumptions of the VC theorem are satisfied.

Properties of $p \mapsto \mu_p$

- **Proposition.**
  - Let $p \in S_G(\mathbb{M})$ be f-generic, and assume that $q \in \overline{Gp}$. Then q is f-generic and $\mu_p = \mu_q$.
  - The map $p \mapsto \mu_p$ is continuous.
- In particular, for every f-generic p there is an almost periodic q such that $\mu_p = \mu_q$.
- We note however that Pillay and Yao give an example of a group definable in an o-minimal theory in which there are weakly generic types that are not almost periodic.

Ergodic measures

- Recall that a G-invariant probability measure μ is *ergodic* if it is an extreme point of the convex set of all G-invariant measures. Equivalently, if for every Borel set Y such that $\mu(Y \triangle gY) = 0$ for all $g \in G$, either $\mu(Y) = 0$ or $\mu(Y) = 1$.

Theorem

[C., Simon] Regular ergodic measures on $S_G(\mathbb{M})$ are precisely the measures of the form μ_p, for f-generic $p \in S_G(\mathbb{M})$.

- In particular, the set of regular ergodic measures is closed.
- **Problem.** Let FGen $\subseteq S_G(\mathbb{M})$ be the closed set of f-generic types, then G/G^{00} acts on FGen. Is the map $(g, p) \mapsto g \cdot p$ measurable? It is continuous for a fixed g and measurable for a fixed p. In many situations this is sufficient for joint measurability, but not so clear in this case.

References

- Artem Chernikov, Pierre Simon, “Definably amenable NIP groups”, arXiv:1502.04365