Higher classification theory and $n$-amalgamation

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1. Tameness notions in Shelah’s classification are typically given by restrictions on the combinatorial complexity of definable binary relations, by forbidding certain induced subgraphs (e.g. $T$ is stable if no definable binary relation can contain arbitrary large finite half-graphs; and NIP if sufficiently large random bipartite graphs are omitted; and distal if bipartite “expanders” are omitted).

2. A typical result then demonstrates that binary relations are “approximated” by the unary ones, up to a “small” error. For example, stationarity of forking in stable theories says that given $p(x), q(y)$ types over a model $M$, there exists a unique type $r(x, y)$ over $M$ so that if $(a, b) \models r$ then $a \models p, b \models q$ and $a \fork_M b$ — that is, there is a unique type $r(x, y)$ extending $p(x) \cup q(y)$, up to the forking formulas $\varphi(x, y) \in \mathcal{L}(M)$. 
1. Another example: $T$ is distal if and only if for any $p(x)$, $q(y)$ global invariant types that commute, there is a unique global type $r(x, y)$ extending $p(x) \cup q(y)$.

2. $T$ is NIP iff for any definable pairwise commuting measures $\mu(x)$, $\nu(y)$, $\varphi(x, y)$ and $\varepsilon > 0$, $\mu \otimes \nu(\varphi(x, y) \Delta \psi(x, y)) < \varepsilon$ for some $\psi(x, y)$ a Boolean combination of $\psi_i(x)$, $\psi'_i(y)$.

3. $n$-tame: any relation $\varphi(x_1, \ldots, x_{n+1})$ can be “approximated” by relations

4. $n$-ary implies $n$-tame for any tameness (1-ary should imply distal - but there are no truly unary theories because of “=”)
We fix a complete theory $T$ in a language $\mathcal{L}$. For $k \geq 1$ we define:

- A formula $\varphi(x; y_1, \ldots, y_k)$ is $k$-dependent if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}$, $i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of $T$ such that $A = \prod_{i=1}^n A_i$ is shattered by $\varphi$, where “$A$ shattered” means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $\mathcal{M} \models \varphi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$.

- $T$ is $k$-dependent if all formulas are $k$-dependent.

- $T$ is strictly $k$-dependent if it is $k$-dependent, but not $(k - 1)$-dependent.

- 1-dependent $= \text{NIP} \subsetneq 2$-dependent $\subsetneq \ldots$, as witnessed e.g. by the theory of the random $k$-hypergraph.
Examples of $n$-dependent structures

Theorem. [C., Hempel] If the field $K$ is NIP, then the theory $T$ of alternating $n$-linear forms over $K$ (generalizing Granger) is (strictly) $n$-dependent.

(And if $K \models ACF$, then $T$ is NSOP$_1$, essentially by the same proof as for $n = 2$ in [C., Ramsey]).

Theorem [Composition Lemma] Let $M$ be an $L'$-structure such that its reduct to a language $L \subseteq L'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \ldots, x_d)$ be an $L$-formula, and $(y_0, \ldots, y_k)$ be arbitrary $k + 1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i^t_1, \ldots, i^t_k \leq k$ be arbitrary, and let $f_t : M_{y_{i^t_1}} \times \ldots \times M_{y_{i^t_k}} \rightarrow M_{x_t}$ be an arbitrary $L'$-definable $k$-ary function. Then the formula

$$\psi(y_0; y_1, \ldots, y_k) := \varphi\left(f_1(y_{i^1_1}, \ldots, y_{i^1_k}), \ldots, f_d(y_{i^d_1}, \ldots, y_{i^d_k})\right)$$

is $k$-dependent.

Our earlier proof for $k = 2$ used a type counting criterion for types over infinite indiscernible sequences, and set-theoretic absoluteness. We have an analogous result for OP$_2$. Also for FOP$_2$ by Abd Aldaim, Conant, Terry.
Proof of the Composition Lemma, 1

- Given a formula $\varphi(x; y_1, \ldots, y_k)$, $\varepsilon \in \mathbb{R}_{>0}$ and a function $f : \mathbb{N} \to \mathbb{N}$, we consider the following condition.

  $(\dagger)_{f, \varepsilon}$ There exists some $n^* \in \mathbb{N}$ such that the following holds for all $n^* \leq n \leq m \in \mathbb{N}$: For any mutually indiscernible sequences $l_1, \ldots, l_k$ of finite length, with $l_i \subseteq M_{y_i}$, $n = |l_1| = \ldots = |l_{k-1}|$, $m = |l_k|$, and $b \in M_x$ an arbitrary tuple there exists an interval $J \subseteq l_k$ with $|J| \geq \frac{m}{f(n)} - 1$ satisfying $|S_{\varphi, J}(b, l_1, \ldots, l_{k-1})| < 2^{n^{k-1}-\varepsilon}$.

- Proposition. The following are equivalent for a formula $\varphi(x; y_1, \ldots, y_k)$, with $k \geq 2$:
  1. $\varphi(x; y_1, \ldots, y_k)$ is $k$-dependent.
  2. There exist some $\varepsilon > 0$ and $d \in \mathbb{N}$ such that $\varphi$ satisfies $(\dagger)_{f, \varepsilon}$ with respect to the function $f(n) = n^d$.
  3. There exist some $\varepsilon > 0$ and some function $f : \mathbb{N} \to \mathbb{N}$ such that $\varphi$ satisfies $(\dagger)_{f, \varepsilon}$.

- This type-counting criterion can then be used to obtain some combinatorial stabilization of shattering on indiscernible arrays:
Proof of the Composition Lemma, 2

(“Kasse II, portato” by Frank Lepold)
Examples of $n$-dependent structures

In some sense all known “algebraic” examples are built from multilinear forms over NIP fields, is there some general theorem like this?

- [Cherlin-Hrushovski] smoothly approximable structures are 2-dependent: coordinatizable by bilinear forms / finite fields,
- infinite extra-special $p$-groups, and strictly $n$-dependent pure groups constructed using Mekler’s construction [C., Hempel], using Baudisch’s interpretation in alternating bilinear maps. Also generic $n$-nilpotent groups of odd prime exponent $p$, d’Elbée, Müller, Ramsey, Siniora.

- Speculation. If $T$ is $n$-dependent, then it is “linear, or 1-based” relative to its NIP part.

- Conjecture. If $K$ is an $n$-dependent field (pure, or with valuation, derivation, etc.), then $K$ is NIP.

- Mounting evidence: $n$-dependent fields are Artin-Schreier closed (Hempel), valued char $p$ are Henselian (C., Hempel), for valued fields reduces to pure fields (Boissonneau),...
Higher amalgamation was studied by a number of authors, starting with Shelah’s work on stability in AEC’s, Hrushovski in the study of the saturation spectrum and of generalized imaginaries, continued in a series of papers by Goodrick, Kim, Kolesnikov and others...

**Definition**

For \( n \in \omega \), let \([n] = \{1, \ldots, n\} \in \omega\). For a set \( X \), we let \( \mathcal{P}(X) \) be the set of all subsets of \( X \), \( \mathcal{P}_{<n}(X) \) (\( \mathcal{P}_{\leq n}(X) \)) the set of all subsets of \( X \) of size less (respectively, less or equal) than \( n \), and \( \mathcal{P}^-(X) := \mathcal{P}(X) \setminus \{X\} \). For \( s \subseteq X \), we let \((\downarrow s) := \{t \subseteq X : t \subseteq s\}\).

We let \( T \) be a complete *simple* first-order theory in a language \( \mathcal{L} \), and we work in \( \mathbb{M}^{\text{heq}} \), the expansion of \( \mathbb{M} \) by the hyper-imaginaries. As usual, \( \downarrow \) denotes forking independence, \( \downarrow^u \) denotes finite satisfiability, and \( bdd(A) \) is the bounded closure of the set \( A \) in \( \mathbb{M}^{\text{heq}} \).
Higher amalgamation, 2

Definition

Let $X$ be an arbitrary small set, and $S \subseteq \mathcal{P}(X)$ be non-empty and closed under subsets (so in particular $\emptyset \in S$). Let $\{r_s(x_s) : s \in S\}$ be a family of complete types over $\emptyset$ (where each $x_s$ is a possibly infinite tuple of variables). We say that such a family of types is independent if:

1. if $a_\emptyset \models r_\emptyset$, then the set of elements of the tuple $a_\emptyset$ is boundedly closed;

2. if $s, t \in S$ and $s \subsetneq t$, then $x_s \subsetneq x_t$ and $r_s \subsetneq r_t$;

3. for all $s, t \in S$ we have $x_s \cap x_t = x_{s \cap t}$;

4. if $s \in S$ and $a_s \models r_s$, then:
   
   4.1 the set $\{a_{\{t\}} : t \in S\}$ is independent over $a_\emptyset$, where $a_{\{t\}}$ is a subtuple of $a_s$ corresponding to the subtuple of the variables $x_{\{t\}} \subseteq x_s$;
   
   4.2 the set of elements of the tuple $a_s$ is equal to $\text{bdd} \left( \bigcup_{t \in S} a_{\{t\}} \right)$, and the map $a_s \to x_s$ between the realizations and the variables is a bijection.
Higher amalgamation, 3

Definition

1. For \( n \geq 1 \), \( T \) satisfies \((\text{independent})\ n\text{-amalgamation}\) if for every independent system of types \( \{r_s(x_s) : s \in \mathcal{P}^{-}([n])\} \) there exists a complete type \( r_n(x_n) \) such that \( \{r_s(x_s) : s \in \mathcal{P}([n])\} \) is an independent system of types.

2. \( T \) satisfies \((\text{independent})\ n\text{-uniqueness}\) if for every independent system of types \( \{r_s(x_s) : s \in \mathcal{P}^{-}([n])\} \) there exists at most one complete type \( r_n(x_n) \) such that \( \{r_s(x_s) : s \in \mathcal{P}([n])\} \) is an independent system of types.

3. \( T \) satisfies \( n\text{-amalgamation (n-uniqueness)}\) \(\text{over a set } A \subseteq \mathcal{M}\) if (1) (respectively, (2)) holds for every independent system of types with \( r_\emptyset = \text{tp}(\text{bdd}(A)) \).

4. \( T \) satisfies \( \text{complete } n\text{-amalgamation (or } \leq n\text{-amalgamation)}\) if \( T \) satisfies \( m\text{-amalgamation} \) for all \( 1 \leq m \leq n \).
Lemma
Assume $n \geq 1$ and $T$ has $(\leq n)$-amalgamation. Assume that $X$ is a set, $s^* \in \mathcal{P}(X)$, $S \subseteq \mathcal{P}_{<n}(X)$ is non-empty and closed under subsets (and if $n = 1$, also that $X = \bigcup \{s : s \in (\downarrow s^*) \cup S\}$), so that $\{r_s(x_s) : s \in (\downarrow s^*) \cup S\}$ is an independent system of types. Then $\{r_s(x_s) : s \in (\downarrow s^*) \cup S\}$ can be extended to an independent system of types $\{r_s(x_s) : s \in \mathcal{P}(X)\}$.

Problem
Is analogous statement true in $\text{NSOP}_1$ theories, with forking independence replaced by Kim-independence? Note that we have used base monotonicity of forking in the proof.
Higher stationarity and $n$-dependence

**Theorem**

Given $n \geq 1$, let $T$ be a simple theory with $\leq (n + 2)$-amalgamation (over models). Then $T$ is $n$-dependent if and only if $T$ has $(n + 1)$-uniqueness (over models).

For $n = 1$ this corresponds to the well-known fact that if $T$ is simple (hence satisfies $\leq 3$-amalgamation over models) and there exists a non-stationary type (i.e. 2-stationarity fails), then $T$ is not NIP.

**Definition (Takeuchi)**

A partitioned formula $\varphi(x; y_1, y_2)$ has OP$_2$ (probably not the final name) if there exist sequences $(a_i)_{i \in \omega}, (b_j)_{j \in \omega}$ with $a_i \in \mathbb{M}^{y_1}, b_j \in \mathbb{M}^{y_2}$ so that for every strictly increasing $f : \omega \to \omega$ there exists $c_f \in \mathbb{M}^x$ satisfying $\models \varphi(c_f, a_i, b_j) \iff i \leq f(j)$ for all $(i, j) \in \omega^2$.

A related property FOP$_2$ with increasing functions replaced by arbitrary functions $f : \omega \to \omega$ was also considered by Takeuchi, and it was studied more recently by Terry and Wolf.
Further notions of binarity

We let $\mathcal{C} := (\mathbb{L}, C)$ be the \textit{generic countable binary branching $C$-relation}, i.e. the Fraïssé limit of all finite binary branching $C$-relations. We also let $\mathcal{C}_\prec := (\mathbb{L}, C, \prec)$ be the \textit{generic countable convexly ordered binary branching $C$-relation}, i.e. the Fraïssé limit of all finite convexly ordered binary branching $C$-relations.

Definition

A theory $T$ is $\mathcal{C}$-less if there is no formula $\varphi(x, y, z)$ and $(a_g : g \in \mathbb{L})$ such that $\models \varphi(a_f, a_g, a_h) \iff \mathcal{C} \models C(f, g, h)$. Equivalently, if every $\mathcal{C}_\prec$-indiscernible is already $(\mathbb{L}, \prec)$-indiscernible.

Related to treeless theories considered by Kaplan, Ramsey, Simon (probably the same).

Theorem

$\mathcal{C}$-less theories form a proper subclass of $\text{NOP}_2$ theories (and more precisely, every $\mathcal{C}$-less formula is $\text{NOP}_2$).
Theorem

If $T$ is simple with $\leq 4$-amalgamation, then the following are equivalent:

1. $T$ satisfies 3-uniqueness;
2. $T$ is 2-dependent;
3. $T$ has no OP$_2$;
4. $T$ has no FOP$_2$;
5. $T$ is $C$-less.

E.g., as bilinear forms over finite fields have a simple theory and satisfy $n$-amalgamation for all $n$, it follows that they are $C$-less.