Convolution Semigroups of measures in NIP groups (joint w/ Kyle Cannon)

\( T \) - a complete \( \mathbb{L} \)-theory

\( \text{IM} = T \) - a monster model of \( T \)

\( S_x(A), \ A \leq \text{IM} \)

\( S_x(\text{IM}) \) - global types

Def. A (Keisler) measure \( \mu \) in vars \( x \) over \( A \leq \text{IM} \) is a finitely additive probability measure on the \( A \)-definable subsets of \( \text{IM}^x \).

\( M_x(A) \) - the compact Hausdorff space of Keisler measures over \( A \) in the variable \( x \), equipped with the topology induced from \( \mathbb{I}^x(A) \) with the product topology.

Basis of open subsets: \( \bigwedge \{ \mu \in M_x(A) : r_i < \mu(\varphi_i(x)) < s_i : \varphi_i(x) \in L(A), r_i, s_i \in [0, 1] \text{ for } i < n \} \) for \( n \in \mathbb{N} \).
Every $\mu \in M^x(A)$ can be viewed as a measure on the open subsets of $S(A)$, then extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S(A)$. Then the topology above corresponds to the weak $*$- topology.

**Convolution products**

Given $p \in S^x(IM)$, $q \in S^y(IM)$, $p \in S^{x+y}(IM)$, then $p \circ q \in S^x(IM)$; for any small $M \leq N < IM$, we let $p \circ q|_N = tp(a, b/N)$ for some $a = q|_N$, $a = p|_N$.

Assume that $T$ expands a group, then given $p, q \in S^x(IM, A)$, we have an invariant type $p * q \in S^x(IM, A)$, via

$v(x) \in p * q \iff v(x \cdot y) \in p \circ q|_y$ for every $v(x) \in CH(M)$.

Equivalently, $p * q = tp(a, b/IM)$, for some $a = p \circ q$. (In some larger model.)
Given \( A \subseteq M \), \( S^\text{inv}_x(M,A) \) — the closed set of global \( A \)-inv. \( x \)-types

\( S^f_{\times}(M,A) \) — (closed) set of global \( f \)-types finitely satisfiable in \( A \)

\[ S^+_x(M,A) \text{ for } f \in \{\text{inv}, f\} \]

\((S^+_x(M,M),*)\) — compact left-continuous semigroup

\[ \forall \left( \forall^+ \subseteq S^+_x(M,M), ^* q : S^+_x(M,M) \right) \text{ is continuous.} \]

Motivation: By a theorem of Newelski, if \( T \) is stable, idempotent \( x \)-types \((S^+_x(M,M),*)\) correspond to type-definable subgroups of \( G \).

Classical case: If \( G \) is a locally compact topological group, the space of regular Borel probability measures on \( G \) is
equipped with the convolution product
\( \mu * \nu (A) = \int \int \chi_A(x, y) \, \mu(x) \, \nu(y) \, d\mu(x) \, d\nu(y) \) for \( A \subseteq G \) Borel.

If \( G \) is compact, then \( \mu \) is idempotent if the support of \( \mu \) is a compact subgroup of \( G \) and \( \mu \) restricted to it is the Haar measure. [Kawada, Itô '40], [Wendel '54].

Same char. extends to locally compact abelian groups [Rudin '59], [Cohen '60].

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**Def.** \( \mu \in M_x(M) \) is Borel definable (over \( M \setminus M \)) if:

1. For any \( \phi(x, y) \in \mathcal{L} \times y \) and \( b \in M_y \), \( \mu(\phi(x, b)) \) depends only on \( f_\phi(b/M) \) (i.e., \( \mu \) is \( M \)-invariant)
2. \( q \in Sy(M) \mapsto \mu(\phi(x, b)) \) for some/any \( b + q \) is Borel.
Given $\mu \in M_x(M)$, $\nu \in M_y(M)$ with $\mu$ Borel def./$M$, $\mu \otimes \nu \in M_{xy}(M)$ via

$$\mu \otimes \nu (\xi(x,y)) = \int M_y(M) \mu(\xi(x,q)) \, d\nu(q)$$

Restrict to NIP groups, i.e., $T$ is an expansion of a group and NIP.

If $\mu, \nu$ are invariant, $\mu \otimes \nu (\xi(x)) = \mu_x \otimes \nu_y (\xi(x,y))$.

Let $M^+_x(M, M)$ for $t \in \xi$ s, inv$^f$.

Fact [c, Gannon] If $T$ is NIP, then $M^+_x(M, M)$ is compact, left-continuous semigroup.
Given \( \mu \in M \times (A) \), \( S(\mu) := \{ p \in S \times (A) : \forall x \in p \Rightarrow \mu(f(x)) > 0 \} \) is the support of \( \mu \).

Not necessarily a group for an idempotent \( \mu \) (e.g., \( M = (S', \cdot, C(x, y, z)) \) — the circle group of rotating circular ordering of the unit circle).

\( \mu \) — the restriction of the Haar measure to definable subsets.

\( S(\mu) \) is not a group.

\((S(\mu), \ast) \cong S' \times \mathbb{R}^+ \times \mathbb{R}^-

\text{Fact [C., Gannon]} Adapting Blickle's work, if \( \mu \in M \times (M) \) is definable, then \((S(\mu), \ast)\) is a compact, left- and right-regular semigroup with no closed two-sided ideals.

\text{Fact [C., Gannon]} If \( T \) is stable, \( \mu \) is any measure.

\text{Fact [C., Gannon]} If \( G \) is abelian, \( \mu \) is g.s.; \( \mu \) is idempotent if \( \mathbb{F} \mu \) is the unique left-invariant measure on the type-def. subgroup \( S \{ \mu \} := \{ g \in M : g \cdot \mu = \mu \} \).
Then \([C., \text{ Gannon}]\).

Assume \(G\) is NIP, let \(I\) be a minimal left ideal of \(M^+ (M, M)\), then:

1) \(I\) is a closed convex subset of \(M^+ (M, M)\).

2) For any \(\mu \in I\), \(\pi^* (\mu) = h\), where \(h\) is the normalized Haar measure on \(G/ G^{00}\) and \(\pi: G \to G/G^{00}\) is the quotient map.

3) For any idempotent \(u \in I\), \(u \ast I\) is trivial.

(In contrast to the case of types, where by the Ellis group conjecture of Newelski (Pillay, if \(G\) is definable, then \(u \ast I \cong G/G^{00}\) - so often non-trivial).

4) Assume \(G\) is \begin{underline}{definably amenable}. \end{underline}

\(\text{In } M^\times (M, M),\) minimal left ideals are of the form

\(I = \xi \circ \beta, \) where \(\beta \in M^\times (M, M)\) is a \(G(M)\)-left-invariant.
5) In $\mathcal{M}^{\text{inv}}_{x}(M, M)$, there exists a unique minimal left (and two-sided) ideal $I = \{ \mu \in \mathcal{M}^{\text{inv}}_{x}(M, M) : \mu \text{ is } G(M)\text{-right-invariant} \}$.

The set $\text{ex}(I)$ is closed (hence $I$ is a Bauer simplex) and equal to $\{ \mu_p : p \in S^{\text{inv}}(M, M) \text{ is right f-generic} \}$.

6) In the fsg, witnessed by a gen-stable $G$-inv meas., the $\{ \mu_3 \}$ is the unique minimal left ideal.

\[ \text{Poulsen Simplex} \]
\[ \rightarrow \text{extremal points are dense.} \]