Towards higher classification theory (n-classification theory)

Shelah's classification: $\underset{\mathbb{M}^x \times \mathbb{M}^y}{R(x,y)}$ - binary relation definable in T.

Tameness assumption: stability, NIP, distality, ... :
certain bipartite finite graphs are omitted in R.

ladder graph — some graph — bipartite expander graphs are forbidden (strong Erdős-Hajnal).

Global
$\Rightarrow$ Conclusion: $R(x,y)$ is "approximated" by unary definable relations $\varphi(x), \psi(y)$.

Ex 1) Stationarity of forking = stability

Given $p(x)$, $q(y)$ types over $M \models T$, there is a unique $r(x,y)$ over M so that if $(a,b) \models r$ then $a \models p$, $b \models q$ and $a \underset{M}{\downarrow} b$: unique $r(x,y)$ extending $p(x) \cup q(y)$, up to forking formulas $\varphi(x,y) \in L(M)$.

2) T is distal $\iff$ for any global inv. $p(x), q(y)$, commuting (i.e. $p \otimes q = q \otimes p$), there is a unique type $r(x,y)$ extending $p(x) \cup q(y)$.

3) T is NIP $\iff$ for any generically stable global measures $\mu(x), \nu(y)$, $\forall \varepsilon > 0$, $\varphi(x,y) \in L(M)$,

$$\mu \otimes \nu(\varphi(x,y)) \approx^{\varepsilon} \mu \otimes \nu \left(\bigsqcup_{i<n} \varphi(x) \wedge \psi(y) \right)$$

for some $\varphi(x), \psi(y), n \in \omega$.

$x \in \mathbb{N}_{\geq 1}$

N-tameness — make a restriction on relations of arity $N+1$, conclude that they are "approx" by relations of arity $\leq N$.

1-NIP = NIP, 1-stable = stable, ...

Best case scenario: T is N-ary : for any $a_1, \dots, a_{N+1}$,

$$\bigcup_{\substack{S \subseteq \{1,\dots,N+1\} \\ |S| = N}} tp((a_i : i \in S)) \vdash tp(a_1, \dots, a_{N+1})$$

Should be N-tame for any notion of N-tame.

E.g. if T is unary, then stable, NIP, distal.

N-dependent theories (N-NIP)

Every definable relation $R(x_1, \dots, x_{n+1})$ omits ~~some~~ finite $(n+1)$-partite $(n+1)$-hypergraph. Equivalently, there are no $A_1, \dots, A_n$ infinite sets, s.t. for every $S \subseteq A_1 \times \dots \times A_n$, $\exists b_S$ s.t. $R(a_1, \dots, a_n, b_S) \iff (a_1, \dots, a_n) \in S$.

T is strictly N-dep if it is N-dep, but not $(N-1)$-dep.

$N = 1 \iff$ NIP.

Thm [C., Hempel] If K is an NIP field, and T is the theory of non-degenerate alternating n-linear forms over K (generalizing Granger), T is strictly N-dependent.

Key lemma Let $\mathcal{M}$ be an L'-structure s.t. its reduct to a language $L \subseteq L'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \dots, x_d)$ be an L-formula, and $(y_0, \dots, y_k)$ a tuple of $k+1$ var's. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \dots, i_k^t \leq k$ be arbitrary, let $f_t : M^{y_{i_1^t}} \times \dots \times M^{y_{i_k^t}} \to M^{x_t}$ be arbitrary

L' - definable k-ary functions. Then the formula

$$\psi(y_0, y_1, \dots, y_k) := \varphi\left(f_1(y_{i_1^1}, \dots, y_{i_k^1}), \dots, f_d(y_{i_1^d}, \dots, y_{i_k^d})\right)$$

is k-dependent.

Ex - smoothly approximable structures [Cherlin-Hrushovski] are 2-dependent

$\boxed{\text{Speculation}}$ If T is n-dependent, then it is "linear" "1-based" over its NIP part.

Conjecture If K is an $\boxed{n\text{-dependent field}}$ (pure, or with valuation, derivation, etc), then K is NIP.

A-S closed val. fields of char p are Henselian

Assume G is a def. group in an N-dep. theory. M is sat, $a_1, \dots, a_{N-1}$, then

$$G^{00}_{M a_1 \dots a_{N-1}} = \bigcap_{i \in \{1, \dots, N-1\}} G^{00}_{M a_1 \dots a_{i-1} a_{i+1} \dots a_{N-1}}$$

$\cap\ \zeta^{\theta\theta}_{N\, a_1,\dots,a_{N-1}}$ for some $|N| \leq_2 |T|^+$.

Thm [C., Towsner] Assume T is k-dep., $k' \geq k+1$, $\mathbb{M} \models T$, $\underline{\mu_1^{(x_1)}, \dots, \mu_{k'}^{(x_{k'})}}$ – global Keisler measures, definable and pairwise commuting (i.e. $\mu_i \otimes \mu_j = \mu_j \otimes \mu_i$). For any $\varphi(x_1, \dots, x_{k'}) \in L(\mathbb{M})$, $\boxed{\varepsilon > 0}$, there exist some $\psi(x_1, \dots, x_{k'})$ a Bool. comb. of formulas each in at most k vars, s.t.

$$\mu_1 \otimes \dots \otimes \mu_{k'} (\varphi \triangle \psi) < \varepsilon.$$

N-distality – <u>Walker</u>

N-stability – Takeuchi, Terry-Wolf, ...