Recognizing groups and fields in Erdős geometry and model theory

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Some practical and structural work in Oaxaca.
The trichotomy principle in model theory: in a sufficiently tame context (certain strongly minimal, o-minimal), every structure is either “trivial”, or essentially a vector space (“modular”), or interprets a field.

Asymptotic sizes of the intersections of definable sets with finite grids in certain model-theoretically tame contexts reflect this trichotomy principle, and detect presence of algebraic structures (groups, fields).

Instances of this principle are well-known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures. Here we discuss “inverse” theorems which show this is the only way.
Sum-product and expander polynomials

- [Erdős, Szemerédi’83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,

$$\max \{|A + A|, |A \cdot A|\} = \Omega \left(|A|^{1+c}\right).$$

- [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any $\varepsilon$).

- [Elekes, Rónyai’00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree $d$, then for all $A, B \subseteq \mathbb{R}$,

$$|f (A \times B)| = \Omega_d \left(n^\frac{4}{3}\right),$$

unless $f$ is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials $g, h, i$. 
Elekes-Szabó theorem

- [Elekes-Szabó’12] provide a conceptual generalization: for any algebraic surface $R(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:

1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq \mathbb{R}$ we have
   \[ |R \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}). \]

2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \rightarrow V$ such that
   \[ \pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \Leftrightarrow R(x_1, x_2, x_3) \]
   for all $x_i \in U_i$. 


Generalizations of the Elekes-Szabó theorem

Let $R \subseteq X_1 \times \ldots \times X_r$ be an algebraic surface (or just a definable set) with finite-to-one projection onto any $r-1$ coordinates and $\dim(X_i) = m$.

1. [Elekes, Szabó’12] $r = 3$, $m$ arbitrary over $\mathbb{C}$ (only count on grids in general position, correspondence with a complex algebraic group of dimension $m$);
2. [Raz, Sharir, de Zeeuw’18] $r = 4$, $m = 1$ over $\mathbb{C}$;
3. [Raz, Shem-Tov’18] $m = 1$, $R$ of the form $f(x_1, \ldots, x_{r-1}) = x_r$ for any $r$ over $\mathbb{C}$.
4. [Hrushovski’13] Pseudofinite dimension, modularity
5. [Bays, Breuillard’18] $r$ and $m$ arbitrary over $\mathbb{C}$, recognized that the arising groups are abelian (no bounds on $\gamma$);
6. Further work: [Wang’15]; [Bukh, Tsimerman’12], [Tao’12]; [Jing, Roy, Tran’19], [Bays, Dobrowolski, Zou].
7. [C., Peterzil, Starchenko] Any $r$ and $m$, any o-minimal structure or stable with a distal expansion and explicit bounds on $\gamma$. A special case:
One-dimensional o-minimal case

Theorem (C., Peterzil, Starchenko)
Assume $r \geq 3$, $M$ is an o-minimal expansion of $\mathbb{R}$ and $R \subseteq \mathbb{R}^r$ is definable, such that the projection of $R$ to any $r-1$ coordinates is finite-to-one. Then exactly one of the following holds.

1. For any finite $A_i \subseteq \mathbb{R}$, $i \in [r]$, we have

\[ |R \cap (A_1 \times \ldots \times A_r)| = O_R \left( n^{r-1-\gamma} \right), \]

where $\gamma = \frac{1}{3}$ if $r \geq 4$, and $\gamma = \frac{1}{6}$ if $r = 3$.

2. There exist open sets $U_i \subseteq \mathbb{R}$, $i \in [r]$, an open set $V \subseteq \mathbb{R}$ containing $0$, and homeomorphisms $\pi_i : U_i \rightarrow V$ such that

\[ \pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \iff R(x_1, \ldots, x_r) \]

for all $x_i \in U_i$, $i \in [r]$. 
General o-minimal case

Theorem (C., Peterzil, Starchenko)

Let \( \mathcal{M} \) be an o-minimal expansion of \( \mathbb{R} \). Assume \( r \geq 3 \), \( R \subseteq X_1 \times \cdots \times X_r \) are definable with \( \text{dim} (X_i) = m \), and the projection of \( R \) to any \( r-1 \) coordinates is finite-to-one. Then exactly one of the following holds.

1. For any finite \( A_i \subseteq_n X_i \) in general position, \( i \in [r] \), we have

\[
|R \cap (A_1 \times \cdots \times A_r)| = O_R \left( n^{r-1-\gamma} \right),
\]

for \( \gamma = \frac{1}{8m-5} \) if \( r \geq 4 \), and \( \gamma = \frac{1}{16m-10} \) if \( r = 3 \).

2. There exist definable relatively open sets \( U_i \subseteq X_i \), \( i \in [r] \), an abelian Lie group \( (G, +) \) of dimension \( m \) and an open neighborhood \( V \subseteq G \) of 0, and definable homeomorphisms \( \pi_i : U_i \rightarrow V \), \( i \in [r] \), such that for all \( x_i \in U_i \), \( i \in [r] \)

\[
\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \iff R(x_1, \ldots, x_r).
\]
Remarks

1. If $\mathcal{M}$ is o-minimal but is not elementarily equivalent to an expansion of $\mathbb{R}$ — only get correspondence with a type-definable group.

2. One ingredient — “Szémeredi-Trotter”-style bounds in o-minimal, and more generally distal structures.

3. Another — a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a “generic chunk”, along with a purely combinatorial version (providing a “coordinatization” result for the associated locally modular pregeometry).
First ingredient: Recognizing groups, 1

1. Assume that \((G, +, 0)\) is an abelian group, and consider the \(r\)-ary relation \(R \subseteq \prod_{i \in [r]} G\) given by \(x_1 + \ldots + x_r = 0\).

2. Then \(R\) is easily seen to satisfy the following two properties, for any permutation of the variables of \(R\):

\[
\forall x_1, \ldots, \forall x_{r-1} \exists! x_r R(x_1, \ldots, x_r), \tag{P1}
\]

\[
\forall x_1, x_2 \forall y_3, \ldots y_r \forall y'_3, \ldots, y'_r \left( R(\bar{x}, \bar{y}) \land R(\bar{x}, \bar{y}') \rightarrow \left( \forall x'_1, x'_2 R(\bar{x}', \bar{y}) \leftrightarrow R(\bar{x}', \bar{y}')) \right) \right), \tag{P2}
\]

We show a converse, assuming \(r \geq 4\):
Recognizing groups, 2

Theorem (C., Peterzil, Starchenko)

Assume \( r \in \mathbb{N}_{\geq 4}, X_1, \ldots, X_r \) and \( R \subseteq \prod_{i \in [r]} X_i \) are sets, so that \( R \) satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group \((G, +, 0_G)\) and bijections \( \pi_i : X_i \rightarrow G \) such that for every \( (a_1, \ldots, a_r) \in \prod_{i \in [r]} X_i \) we have

\[
R(a_1, \ldots, a_r) \iff \pi_1(a_1) + \ldots + \pi_r(a_r) = 0_G.
\]

If \( X_1 = \ldots = X_r \), property (P1) is equivalent to saying that the relation \( R \) is an \((r - 1)\)-dimensional permutation on the set \( X_1 \), or a Latin \((r - 1)\)-hypercube, as studied by Linial and Luria. Thus the condition (P2) characterizes, for \( r \geq 3 \), those Latin \( r \)-hypercubes that are given by the relation \( x_1 + \ldots + x_{r-1} = x_r \) in an abelian group.

If \( R \) is definable and \( X_i \) are type-definable in a (saturated) \( M \), then \( G \) is type-definable and \( \pi_i \) are relatively definable in \( M \).
Recognizing groups in the stable case

- In the stable version of our theorem, we only get “generic correspondence” with a type-definable group.
- An \( r \)-gon is a tuple \( a_1, \ldots, a_r \) such that any \( r - 1 \) of its elements are (forking-)independent, and any element in it is in the algebraic closure of the other ones.
- An \( r \)-gon is abelian if, after any permutation of its elements, we have \( a_1a_2 \downarrow_{\text{acl}(a_1a_2) \cap \text{acl}(a_3 \ldots a_r)} a_3 \ldots a_r \).
- If \((G, \cdot)\) is a type-definable abelian group, \( g_1, \ldots, g_{r-1} \) are independent generics in \( G \) and \( g_r := g_1 \cdot \ldots \cdot g_{r-1} \), then \( g_1, \ldots, g_r \) is an abelian \( r \)-gon (associated to \( G \)).
- Conversely,

**Theorem (C., Peterzil, Starchenko; independently Hrushovski)**

Let \( r \geq 4 \) and \( a_1, \ldots, a_r \) be an abelian \( r \)-gon. Then there is a type-definable (in \( \mathcal{M}^{\text{eq}} \)) connected abelian group \((G, \cdot)\) and an abelian \( r \)-gon \( g_1, \ldots, g_s \) associated to \( G \), such that after a base change each \( g_i \) is interalgebraic with \( a_i \).
Second ingredient: distality

Definition
A structure $\mathcal{M}$ is distal if and only if for every definable family $\{\varphi(x, b) : b \in M_y\}$ of subsets of $M_x$ there is a definable family $\{\theta(x, c) : c \in M_y^k\}$ such that for every $a \in M_x$ and every finite set $B \subset M_y$ there is some $c \in B^k$ such that:

1. $a \models \theta(x, c)$;
2. $\theta(x, c) \vdash \text{tp}_\varphi(a/B)$, that is for every $a' \models \theta(x, c)$ and $b \in B$ we have $a' \models \phi(x, b) \iff a \models \phi(x, b)$.
Examples of distal structures

- \( M \) distal \( \implies M \) is NIP, unstable.

- Examples of distal structures: (weakly) \( o \)-minimal structures, various valued fields of char 0 (e.g. \( \mathbb{Q}_p \), RCVF, the valued differential field of transseries).

- Stable structures with distal expansions: \( \text{ACF}_0 \), \( \text{DCF}_{0,m} \), \( \text{CCM} \), abelian groups, Hrushovski constructions*.

- Stable structures without distal expansions: \( \text{ACF}_p \) [C., Starchenko’15], a disjoint union of finite expander graphs (e.g. Ramanujan graphs) of growing degree and expansion [Jiang, Nešetřil, Ossona de Mendez, Siebertz’20], any Banach space [Hanson] (distality in continuous logic is developed by [Anderson]).

- **Problem.** Do non-abelian free groups have distal expansions?
Number of edges in a $K_{k,\ldots,k}$-free hypergraph

The following fact is due to [Kővári, Sós, Turán’54] for $r = 2$ and [Erdős’64] for general $r$.

Fact (The Basic Bound)

If $H$ is a $K_{k,\ldots,k}$-free $r$-hypergraph then $|E| = O_{r,k} \left( n^{r-\frac{1}{kr-1}} \right)$.

So the exponent is slightly better than the maximal possible $r$ (we have $n^r$ edges in $K_{n,\ldots,n}$). A probabilistic construction in [Erdős’64] shows that it cannot be substantially improved.
Bounds for graphs definable in distal structures

- Generalizing [Fox, Pach, Sheffer, Suk, Zahl’15] in the semialgebraic case, we have:

**Fact (C., Galvin, Starchenko’16)**

Let $\mathcal{M}$ be a distal structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists some $\varepsilon = \varepsilon(R, k) > 0$ such that for any $A_1 \subseteq_n M_{x_1}, A_2 \subseteq_n M_{x_2}$, if $E := R \cap (A_1 \times A_2)$ is $K_{k,k}$-free then $|E| = O_{R,k}(n^{t-\varepsilon})$, where $t$ is the exponent given by the Basic Bound for arbitrary graphs.

- In fact, $\varepsilon$ is given in terms of $k$ and the size of the smallest distal cell decomposition for $R$.
- E.g. if $R \subseteq M^2 \times M^2$ for an o-minimal $\mathcal{M}$, then $t - \varepsilon = \frac{4}{3}$ ([C., Galvin, Starchenko’16]; independently, [Basu, Raz’16]).
Recognizing fields

- For the semialgebraic $K_{2,2}$-free point-line incidence relation $R = \{ (x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1 x_1 + y_2 \} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have the (optimal) lower bound $|R \cap (V_1 \times V_2)| = \Omega(n^{3/4})$.
- To define it we use both addition and multiplication, i.e. the field structure.
- This is not a coincidence — any non-trivial lower bound on the Zarankiewicz exponent of $R$ allows to recover a field from it:

Theorem (Basit, C., Starchenko, Tao, Tran)

Assume that $\mathcal{M} = (M, <, \ldots)$ is o-minimal and $R \subseteq M_{x_1} \times \ldots \times M_{x_r}$ is a definable relation which is $K_{k, \ldots, k}$-free, but $|R \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})$ for $V_i \subseteq_n M_{x_i}$. Then a real closed field is definable in the first-order structure $(M, <, R)$. 
Ingredients

- An (almost) optimal bound on the number of edges in $K_{k,...,k}$-free hypergraphs definable in locally modular o-minimal expansions of groups, so e.g. for semilinear (= definable in $(\mathbb{R}, <, +)$) hypergraphs.
- The trichotomy theorem for o-minimal structures [Peterzil, Starchenko’98].
Local modularity

- We write \( a \triangleleft_C B \) to denote that \( \dim(a/BC) = \dim(a/C) \) in the matroid associated to the algebraic closure in an \( o \)-minimal structure.

- An \( o \)-minimal structure is (weakly) \emph{locally modular} if for any small subsets \( A, B \subseteq M \models T \) there exists some small set \( C \triangleleft_{\emptyset} AB \) such that \( A \triangleleft_{\acl(AC) \cap \acl(BC)} B \).

- Intuition: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.

- In particular, an \( o \)-minimal structure is locally modular if and only if any normal interpretable family of plane curves in \( T \) has dimension \( \leq 1 \).
Bound for semilinear relations

Theorem (Basit, C., Starchenko, Tao, Tran)
Let \( \mathcal{M} \) be an o-minimal locally modular expansion of a group and \( Q \) a definable relation of arity \( r \geq 2 \). Then for any \( \varepsilon > 0 \) and any \( V_i \) with \( |V_i| = n \) such that \( E := Q \cap V_1 \times \ldots \times V_r \) is \( K_{k,\ldots,k} \)-free, we have
\[
|E| = O_{Q,k,\varepsilon}\left(n^{r-1+\varepsilon}\right).
\]
Moreover, if \( Q \) itself is \( K_{k,\ldots,k} \)-free, then for any \( V_i \) with \( |V_i| = n \) we have
\[
|E| = O_Q(n^{r-1}).
\]
Recovering a field in the o-minimal case

Fact (Peterzil, Starchenko’98)

Let $\mathcal{M}$ be an o-minimal (saturated) structure. TFAE:

- $\mathcal{M}$ is not locally modular;
- there exists a real closed field definable in $\mathcal{M}$.

- [Marker, Peterzil, Pillay’92] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\cdot \upharpoonright_{[0,1]^2}$ is definable in $(\mathbb{R}, <, +, X)$. In particular, it is not locally modular.

- Combining this with the “optimal” bound in the locally modular case, we get the result.

- Problem: is it possible to establish a more direct correspondence between the relation with many edges and the point-line incidence relation in a field?
An application to incidences with polytopes

Applying with $r = 2$ we get the following:

Corollary

For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.

Let $d \in \mathbb{N}$ and $H_1, \ldots, H_s \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_s$.

For any set $V_1$ of $n_1$ points in $\mathbb{R}^d$ and any set $V_2$ of $n_2$ polytopes in $\mathcal{F}$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $\alpha n (\log n)^s$ incidences.

In particular (this corollary was obtained independently by [Tomon, Zakharov]):

Corollary

For any set $V_1$ of $n_1$ points and any set $V_2$ of $n_2$ (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k} \left(n(\log n)^{2d}\right)$ incidences.
Dyadic rectangles and a lower bound

- Is the logarithmic factor necessary?
- Using a different argument, restricting to dyadic boxes on the plane \((d = 2)\), we gave a stronger upper bound \(O\left(n\frac{\log n}{\log \log n}\right)\), and give a construction showing a matching lower bound (up to a constant).
- [Tomon, Zakharov] use our construction to disprove a conjecture of Alon, Basavaraju, Chandran, Mathew, and RajendraPrasad regarding the maximal possible number of edges in a graph of bounded separation dimension.
- We asked what happens for higher \(d\)?
- Very recently, [Har-Peled, Chan] improve our upper bound to \(O\left(n\left(\frac{\log n}{\log \log n}\right)^{d-1}\right)\), and point out that a result of [Chazelle’90] on “data structures” shows its optimality for all \(d\)!
Congrats Kobi and Sergei!