Measures in model theory

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Logic Colloquium 2021, Poznań (via Zoom)

Jul 24, 2021

Spaces of types

- Let T be a complete first-order theory in a language $\mathcal{L}$, $\mathbb{M} \models T$ a monster model (i.e. κ-saturated and κ-homogeneous for a sufficiently large cardinal κ), $\mathcal{M} \preceq \mathbb{M}$ a small elementary submodel.
- For $A \subseteq \mathbb{M}$ and x an arbitrary tuple of variables, $S_x(A)$ denotes the set of complete types over A.
- Let $\mathcal{L}_x(A)$ denote the set of all formulas $\varphi(x)$ with parameters in A, up to logical equivalence — which we identify with the Boolean algebra of A-definable subsets of $\mathbb{M}_x$; $\mathcal{L}_x := \mathcal{L}_x(\emptyset)$.
- Then the types in $S_x(A)$ are the ultrafilter on $\mathcal{L}_x(A)$.
- By Stone duality, $S_x(A)$ is a totally disconnected compact Hausdorff topological space with a basis of clopen sets of the form
$$\langle\varphi\rangle := \{p \in S_x(A) : \varphi(x) \in p\}$$
for $\varphi(x) \in \mathcal{L}_x(A)$.
- We refer to types in $S_x(\mathbb{M})$ as *global types*.

Keisler measures

- A *Keisler measure* μ in variables x over $A \subseteq \mathbb{M}$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of A-definable subsets of $\mathbb{M}_x$.
- $\mathfrak{M}_x(A)$ denotes the set of all Keisler measures in x over A.
- Then $\mathfrak{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

  with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A), r_i, s_i \in [0,1]$ for $i < n$.
- Identifying p with the Dirac measure δ_p, $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathfrak{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$.

Some examples of Keisler measures, 1

- In arbitrary T, given $p_i \in S_x(A)$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ with $\sum_{i\in\mathbb{N}} r_i = 1$, $\mu := \sum_{i\in\mathbb{N}} r_i \delta_{p_i} \in \mathfrak{M}_x(A)$.
- Let $T = \mathrm{Th}(\mathbb{N}, =)$, $|x| = 1$. Then

$$S_x(\mathbb{M}) = \{\mathrm{tp}(a/\mathbb{M}) : a \in \mathbb{M}\} \cup \{p_\infty\},$$

  where p_∞ is the unique non-realized type axiomatized by $\{x \neq a : a \in \mathbb{M}\}$. By QE, every formula is a Boolean combination of $\{x = a : a \in \mathbb{M}\}$, from which it follows that every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is as in (1).
- More generally, if T is ω-stable (e.g. strongly minimal, say ACF_p for p prime or 0) and x is finite, then every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is a sum of types as in (1).
- Let $T = \mathrm{Th}(\mathbb{R}, <)$, λ be the Lebesgue measure on $\mathbb{R}$ and $|x| = 1$. For $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, define $\mu(\varphi) := \lambda\left(\varphi(\mathbb{M}) \cap [0,1]_{\mathbb{R}}\right)$ (this set is Borel by QE). Then $\mu(X)$ is a Keisler measure, but not a sum of types as in (1).

Some examples of Keisler measures, 2

- Let $\mathcal{M} = \prod_{i \in \omega} \mathcal{M}_i / \mathcal{U}$ for some finite $\mathcal{M}_i$ and $\mathcal{U}$ a non-principal ultrafilter on ω. For $\varphi(x, a) \in \mathcal{L}_x(\mathcal{M})$ with $a = (a_i : i \in \omega)/\mathcal{U}, a_i \in M_i$, define

$$\mu(\varphi(x, a)) := \lim_{\mathcal{U}} \frac{|\varphi(M_i, a_i)|}{|M_i|}.$$

  Then μ is a Keisler measure over $\mathcal{M}$.

Brief history of the theory of Keisler measures

- Measures and forking in stable/NIP theories [Keisler'87]
- Automorphism-invariant measures in ω-categorical structures [Albert'92, Ensley'96]
- Applications to neural networks [Karpinski, Macyntire'00]
- Pillay's conjecture and compact domination [Hrushovski, Peterzil, Pillay'08], [Hrushovski, Pillay'11], [Hrushovski, Pillay, Simon'13]
- Randomizations [Ben Yaacov, Keisler'09] (NIP and stability are preserved)
- Approximate Subgroups [Hrushovski'12]
- Definably amenable NIP groups [C., Simon'15] (in particular translation-invariant measures are classified)
- Tame (equivariant) regularity lemmas: subsets of [C., Conant, Malliaris, Pillay, Shelah, Starchenko, Terry, Tao, Towsner, ...'11– ...]
- See my review "Model theory, Keisler measures and groups", The Bulletin of Symbolic Logic, 24(3), 336-339 (2018)

Model theoretic tameness and (hyper-)graph regularity

- Classification theory: Shelah's dividing lines express limitations on definable binary relations, by forbidding certain finitary combinatorial configurations (stability, NIP, simplicity, ...).
- Often on the tame case, obtain consequences of the form: types (over infinite sets) in more than one variable are controlled by unary types, up to a "small error" (e.g. stationarity of non-forking in stable theories, up to algebraic closure).
- Generalizations of these results to Keisler measures provide variants of the celebrated Szemerédi's regularity lemma in combinatorics (about the "generic", or typical, behavior of large *finite* graphs).
- More precisely, the "analytic" presentation of the regularity lemma ([Elek-Szegedy], [Tao], [Towsner], . . .):

Szemerédi's regularity lemma

Theorem

For every $\varepsilon \in \mathbb{R}_{>0}$ *there exists* $K = K(\varepsilon) \in \mathbb{N}$ *s.t.: for any structure* $\mathcal{M}$*, definable relation* $E(x_1, x_2)$ *and Keisler measures* μ_i *on* M_{x_i} *(satisfying a Fubini assumption that always holds for ultraproducts of finite measures), there are definable partitions* $M_{x_i} = \bigsqcup_{j<K} A_{i,j}$ *and* $\Sigma \subseteq \{1, \ldots, K\}^2$ *such that:*

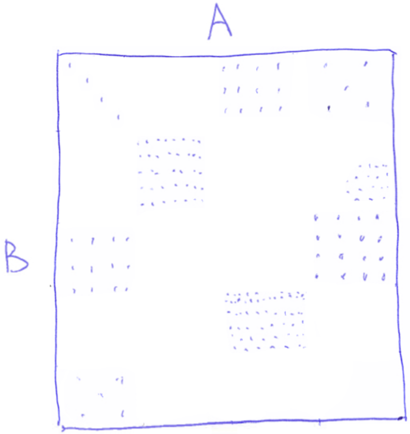
1. $\mu\left(\bigcup_{(i_1,i_2)\in\Sigma} A_{1,i_1} \times A_{2,i_2}\right) \leq \varepsilon$*, where* $\mu = \mu_1 \otimes \mu_2$*,*
2. *for all* $\vec{i} = (i_1, i_2) \notin \Sigma$ *and definable* $A'_i \subseteq A_i$ *we have*

$$\left|\mu\left(E \cap (A'_{1,i_1} \times A'_{2,i_2})\right) - \delta_{\vec{i}}\mu(A'_{1,i_1} \times A'_{2,i_2})\right| < \varepsilon\mu(A_{1,i_1} \times A_{2,i_2})$$

for $\delta_{\vec{i}} = \frac{\mu(E\cap A_{1,i_1}\times A_{2,i_2})}{\mu(A_{1,i_1}\times A_{2,i_2})}$.

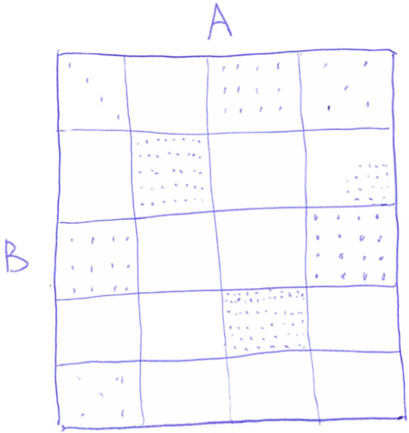
Szemeredi's regularity, 1

- Consider the incidence matrix of a bipartite graph $E \subseteq A \times B$:



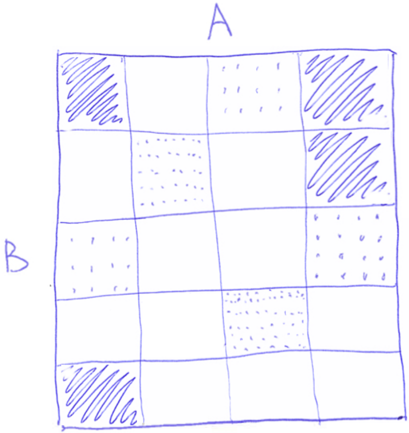
Szemeredi's regularity, 2

- Consider the incidence matrix of a bipartite graph $E \subseteq A \times B$:



Szemeredi's regularity, 3

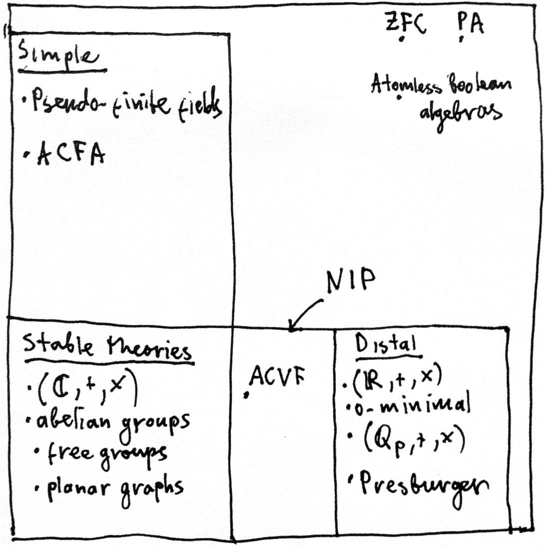
- Consider the incidence matrix of a bipartite graph $E \subseteq A \times B$:



Variants and limitations

- Generalization to hypergraphs [Nagle, Rödl, Schacht], [Rödl, Skokan], [Gowers].
- Some features for general graphs:
  - [Gowers] $K(\varepsilon)$ grows as an exponential tower of 2's of height polynomial in $\frac{1}{\varepsilon}$;
  - Bad pairs are unavoidable in general (half-graphs);
  - Quasi-randomness (intermediate densities) is unavoidable in general.
- Turns out some of the dividing lines in Shelah's classification provide an explanation for these phenomena.

Model theoretic classification



- See ForkingAndDividing.com for an interactive version.

Regularity lemma for NIP relations

Theorem (C., Starchenko)

Let $\mathcal{M}$ be an NIP structure and $k \geq 2$. For every definable relation $E(x_1, \ldots, x_k)$ there is some $c = c(E)$ such that for any $\varepsilon > 0$ and Keisler measures μ_i on M_{x_i} satisfying Fubini there are partitions $M_{x_i} = \bigcup_{j<K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \ldots, K\}^k$ such that:

1. $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
2. $\mu\left(\bigcup_{(i_1,\ldots,i_k)\in\Sigma} A_{1,i_1} \times \ldots \times A_{k,i_k}\right) \leq \varepsilon$, *where* $\mu = \mu_1 \otimes \ldots \otimes \mu_k$,
3. *for all* $\vec{i} = (i_1, \ldots, i_k) \notin \Sigma$ *we have*

$$\left|\mu\left(E \cap (A_{1,i_1} \times \ldots \times A_{k,i_k})\right) - \delta_{\vec{i}}\mu(A_{1,i_1} \times \ldots \times A_{k,i_k})\right| < \varepsilon\mu(A_{1,i_1} \times \ldots \times A_{k,i_k})$$

   for some $\delta_{\vec{i}} \in \{0, 1\}$.

4. *each $A_{i,j}$ is defined by an instance of an E-formula depending only on E and ε.*

Regularity lemma for NIP relations, continued

- Relies on the close connection of NIP and the Vapnik-Chervonenkis, or VC, theory (e.g. existence of ε-nets).
- Generalizes the earlier work in the binary case (i.e. $k = 2$) by [Alon, Fischer, Newman], [Lovász, Szegedy].
- If $\mathcal{M}$ is stable, then in addition (generalizing [Malliaris, Shelah] in the binary case):
  1. we can take the μ_i's to be arbitrary Keisler measures (the Fubini condition is automatically satisfied),
  2. we may assume that $\Sigma = \emptyset$, i.e. all tuples in the partition are ε-regular.
- If $\mathcal{M}$ is distal, then in addition (generalizing [Fox, Pach, Suk] in the semialgebraic case):
  1. for all $(i_1, \ldots, i_k) \notin \Sigma$, either $(A_{1,i_1} \times \ldots \times A_{k,i_k}) \cap E = \emptyset$ or $A_{1,i_1} \times \ldots \times A_{k,i_k} \subseteq E$,
  2. if the relation E is defined by an instance of a formula θ, then we can take each $A_{i,j}$ to be defined by an instance of a formula $\psi_i(x_i, z_i)$ which only depends on θ (and not on ε).

Definably amenable groups

Definition

Let G be a definable group in some structure (i.e. the set of its elements and the group operation are definable).

- A measure μ on the definable subsets of G is (left) *G-invariant* if $\mu(X) = \mu(g \cdot X)$ for all definable $X \subseteq G$ and $g \in G$.
- G is *definably amenable* if there exists a G-invariant Keisler measure on definable subsets of G.

- Note: there exists a left-invariant measure iff exists a right invariant measure; definable amenability is preserved under elementary equivalence.

Examples of definably amenable groups

- Solvable groups, or more generally any group G such that $G(M)$ is amenable as a discrete group.
- Definable compact groups in o-minimal theories or in p-adics (compact Lie groups, e.g. $SO(3, \mathbb{R})$, seen as definable groups in $\mathbb{R}$).
- Stable groups (in particular the free group F_2, viewed as a structure in a pure group language, is definably amenable).
- Ultraproducts of finite groups.
- But: $SL(n, \mathbb{R})$ is not definably amenable for $n > 1$.

Definable amenability in NIP groups, 1

- The theory of definably amenable NIP groups was developed in the last decade, and played an important role in the proof of Pillay's conjecture for groups in *o*-minimal theories [Hrushovski, Peterzil, Pillay].
- [Shelah] If G is NIP, then there exists the smallest type-definable subgroup G^{00} of G of bounded index.
- The quotient G/G^{00} is equipped with the logical topology: a set is closed if its preimage in G is type-definable.
- With this topology G/G^{00} is a compact topological group, hence carries the Haar measure h.
- Example: if $G = \mathrm{SO}(2, \mathcal{R})$ is the circle group defined in a (saturated) real closed field $\mathcal{R}$, then G^{00} is the set of infinitesimal elements of G and G/G^{00} is isomorphic to the standard circle group $\mathrm{SO}(2, \mathbb{R})$.

Definable amenability in NIP groups, 2

- The assumption of definable amenability in NIP allows to recover some ideas of stable group theory, including a theory of generic sets (with connections to topological dynamics following [Newelski]), which leads to a proof of the Ellis group conjecture [C., Simon].

Theorem (C., Simon)

Ergodic measures on G are precisely the "liftings" of the Haar measure on G/G^{00}

$$\mu_p(\varphi(x)) := h\left(\left\{\bar{g} \in G/G^{00} : \varphi(x) \in g \cdot p\right\}\right)$$

for some f-generic type $p \in S_G(\mathbb{M})$.

- (Partial) development of this theory "locally" leads to further applications combining the two lines: regularity lemmas *in groups*, approximating sets by cosets instead of arbitrary sets up to an error of small measure [Terry, Wolf], [Conant, Pillay, Terry].

Keisler measures outside of NIP

- All of the above — inside the context of NIP theories (thanks to the (equivariant) *VC-theory*, measures are strongly approximated by types). What happens in simple theories?
- Ultraproducts of finite counting measures in pseudofinite fields are very well-behaved, e.g. manifested in a strong regularity lemma for definable graphs [Tao].
- But very few general results outside of NIP so far. Some counterexamples:
  - Independent product $\otimes$ of Borel-definable measures is not associative in general [Conant, Gannon, Hanson'21];
- And some positive results:
  - A generalization of ε-nets for n-dependent theories, and the corresponding regularity lemma approximating relations of any arity by relations of arity n [C.,Towsner] (the case $n = 1$ corresponds to the NIP case discussed above).
  - NSOP_1 is preserved under Keisler randomizations [Ben Yaacov, C., Ramsey, 21+]

Definable amenability for groups in simple theories

- Pillay: are there groups definable in simple theories that are not definably amenable?
- (Earlier, Harrington asked a variant of this question with respect to the automorphism group invariance/forking.)
- Note: in the main examples of simple theories, e.g. pseudo-finite fields or ACFA, all groups are definably amenable (typically either pseudo-finite or solvable).

Tarski's characterization of amenability

- A *paradoxical decomposition* for a discrete group G consists of pairwise disjoint subsets $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ of G for some $m, n \in \mathbb{N}_{\geq 1}$ and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that G is the union of the $g_i X_i$ and is also the union of the $h_j Y_j$.
- [Tarski] G is amenable if and only if G has no paradoxical decomposition.

An analog for definable amenability, 1

- We fix a definable group G in a structure M.
- By an *(m-)cycle* (for $m \geq 0$) we mean a formal sum $\sum_{i=1,\ldots,m} X_i$ of definable subsets X_i of G. If all the X_i are the same we could write this formal sum as mX_i. We can add such cycles in the obvious way to get the "free abelian monoid" generated by the definable subsets of G. And any definable subset X of G (including G itself) is a (1-)cycle.
- If $X = \sum_{i=1,\ldots,m} X_i$ and $Y = \sum_{j=1,\ldots,n} Y_j$ are two cycles, then by a *definable piecewise* translation f from X to Y we mean a map f from the formal disjoint union $X_1 \sqcup \ldots \sqcup X_m$ to the formal disjoint union $Y_1 \sqcup \ldots \sqcup Y_n$ for which there is a partition of each X_i into definable subsets $X_{i1}, \ldots, X_{in_i}$, and for each i and $t \leq n_i$, an element g_{it} of G such that the restriction $f|X_{it}$ of f to X_{it} is just left translation by g_{it}, and $g_{it}X_{it}$ is a subset of one of the Y_j's.
- A definable piecewise translation f is said to be *injective* if it is injective as a map between formal disjoint unions.

An analog for definable amenability, 2

- We write $X \leq Y$ if there is an injective piecewise definable translation f from X to Y. Note that $\leq$ is reflexive and transitive. Also $X \leq W$ and $Y \leq Z$ implies $X + Y \leq W + Z$.

Definition

By a *definable paradoxical decomposition* of the definable group G we mean an injective definable piecewise translation from $G + Y$ to Y for some cycle Y.

Theorem (Hrushovski, Pillay)

G is definably amenable if and only if G does not have a definable paradoxical decomposition.

- [Corollary] G is not definably amenable iff $(n+1)G \leq nG$ for some $n \geq 1$.
- Tarski's condition corresponds to: $2G \leq G$. It is open if we can always take $n = 2$ in the definable case.

Theorem (C., Hrushovski, Kruckman, Krupinski, Moconja, Pillay and Ramsey'21)

Let T be a model complete theory eliminating $\exists^\infty$ and G a definable group in T. Assume that (in some model) G contains a (not necessarily definable) free group on ≥ 2 generators. Then there exists a model complete expansion T^* of T so that G is not definably amenable in T^*, and so that if T is simple, then T^* is also simple.

- Example: start with $G := SL_2(\mathbb{C})$ definable in the stable theory ACF_0, obtain a simple (SU-rank 1) theory with a non-definably amenable group.
- The expansion is obtained by adding a "generic" paradoxical decomposition to G. Some interesting tree combinatorics is required to demonstrate that it is axiomatizable, and an explicit description of forking in T^* is obtained in terms of T.

A new journal:

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