Combinatorial properties of non-archimedean convex sets

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Convexity in valued fields

- Introduced by Monna in 1940's, extensively studied in non-archimedean functional analysis.
- **Notation.** K a valued field (e.g. $\mathbb{Q}_p$), with value group $\Gamma = \Gamma_K$, valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue field $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$.
- For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is *convex* if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \ldots, x_n \in X$, and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ we have $\alpha_1 x_1 + \ldots + \alpha_n x_n \in X$ (in the vector space K^d).
- The family of convex subsets of K^d will be denoted Conv_{K^d}.

Convex combinations

- Given an arbitrary set $X \subseteq K^d$, its *convex hull* $\mathrm{conv}(X)$ is the convex set given by the intersection of all convex sets containing X, equivalently the set of all *convex combinations* from X:

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, \alpha_i \in \mathcal{O}, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

- **Prop.** Let K be a valued field and $X \subseteq K^d$. If X is closed under 3-element convex combinations (in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$), then X is convex.
- **Prop.** 2-element convex combinations suffice iff $k \ncong \mathbb{F}_2$.

Convex subsets of $\mathbb{R}^n$ vs convex subsets of K^n

- Parallel: combinatorics of convex subsets of $\mathbb{R}^n$ vs definable subsets of $\mathbb{R}^n$ vs. definable subsets of $\mathbb{Q}_p$.
- **Example** (Marker). Naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0,1] \to [0,1]$ such that

$$C := \{(x, y) : x \in [0,1], 0 \leq y \leq f(x)\}$$

  is convex but the set of points where f is not differentiable is exactly $\left\{\frac{1}{n} : n \in \mathbb{N}_{\geq 2}\right\}$. Now in the field of reals with a predicate for C we can define f and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.
- In contrast, turns out that convex sets in K^n are tame both model theoretically and combinatorially, so we get the best of both worlds.

Convex subsets and $\mathcal{O}$-submodules of K^d

- **Prop.** Nonempty convex subsets of K^d are precisely the translates of $\mathcal{O}$-submodules of K^d.
- **Proof.** First, $\mathcal{O}$-submodules of K^d are clearly convex and contain 0. Conversely, suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1 - \alpha) 0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore C is an $\mathcal{O}$-submodule. And set can be translated to contain 0 (affine maps preserve convexity).
- From this, easy to see that the convex subsets of $K = K^1$ are exactly $\emptyset$ and the quasi-balls (i.e. sets $B = \left\{x \in K^d : \nu(x - c) \in \Delta\right\}$ for some $c \in K$ and an upwards closed subset Δ of Γ_∞).

Algebraic description of convex sets

- **Def.** A valued field K is *spherically complete* if every nested family of (closed or open) valuational balls has non-empty intersection.
- **Thm.** Suppose K is a spherically complete valued field, $d \in \mathbb{N}_{\geq 1}$, and let $C \subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and a decreasing sequence of nonempty, upwards-closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ such that $C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$.

Further properties of this presentation

- $\Delta_d = \{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C\}$. That is, Δ_d is the quasi-radius of the largest quasi-ball around 0 contained in C.
- F_{d-1} can be chosen to be *any* linear hyperplane H in K^d such that every element of C differs from an element of H by a vector in K^d with valuation in Δ_d.
- **Cor.** If K is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the non-empty convex subsets of K^d are precisely the affine images of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.
- By contrast to Marker's example: if K is a spherically complete, then every convex subset of K^d is definable in the expansion of the field K by a predicate for each Dedekind cut of the value group (definable in *Shelah expansion* of K by externally definable sets, so e.g. NIP if K was). In particular, if K has value group $\mathbb{Z}$, then all convex subsets of K^d form a definable family.

Combinatorial consequences

- Using this (combinatorial properties below pass to spherical completions), we can get:
- **Thm.** Let K be a valued field and $d \geq 1$. Then the family Conv_{K^d} has *breadth* d. That is, any nonempty intersection of finitely many convex subsets of K^d is the intersection of at most d of them. (Not true for convex subsets of $\mathbb{R}^2$!)
- **Cor.** The *Helly number* of Conv_{K^d} is $d+1$. I.e., given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every $(d+1)$-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \emptyset$.)
- **Cor.** Conv_{K^d} has VC-dimension $d+1$ and dual VC-dimension d.

Fractional Helly Property

- Combining this with Matoušek's theorem, we obtain:
- **Cor.** The *fractional Helly number* of the family Conv_{K^d} is at most $d+1$ (exactly $d+1$ if K is infinite). I.e. for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$ so that: for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathrm{Conv}_{K^d}$ (possibly with repetitions), if there are $\geq \alpha\binom{n}{d+1}$ $(d+1)$-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a non-empty intersection, then there are $\geq \beta n$ sets from $\{S_1, \ldots, S_n\}$ with a non-empty intersection.
- Moreover, β can be chosen depending only on d and α (and not on the field K).

- Finally, combining these, we obtain an analog of the Boros-Füredi/Bárány selection lemma over valued fields (answering a question of Peterzil and Kaplan):
- **Thm.** For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that: for any valued field K and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of X of size $d + 1$.