Regularity for slice-wise stable hypergraphs

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Joint work with Henry Towsner (U Penn).
For each $i \in \mathbb{N}$, let $G_i = (V_i, E_i)$ be a graph with $|V_i|$ finite and $\lim_{i \to \infty} |V_i| = \infty$.

Given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultraproduct

$$(V, E) = \prod_{i \in \mathbb{N}} (V_i, E_i)$$

is a graph on the set $V$ of size continuum.

Given $k \in \mathbb{N}$ and an internal set $X \subseteq V^k$ (i.e. $X = \prod_{i \in \mathbb{U}} X_i$ for some $X_i \subseteq V_i^k$), we define $\mu^k(X) := \lim_{\mathcal{U}} \frac{|X_i|}{|V_i|^k}$. Then:

- $\mu^k$ is a finitely additive probability measure on the Boolean algebra of internal subsets of $V^k$,
- extends uniquely to a countably additive measure on the $\sigma$-algebra $\mathcal{B}_k$ generated by the internal subsets of $V^k$. 

Context: ultraproducts of finite graphs with Loeb measure
Let $\mathcal{B}_1 \times \mathcal{B}_1$ be the product $\sigma$-algebra, i.e. for every $E \in \mathcal{B}_1 \times \mathcal{B}_1$ and $\varepsilon > 0$ there exist $A_i, B_i \in \mathcal{B}_1$, $i < k$, so that

$$
\mu^2 \left( E \Delta \left( \bigcup_{i<k} A_i \times B_i \right) \right) < \varepsilon.
$$

Note: $\mathcal{B}_1 \times \mathcal{B}_1 \subsetneq \mathcal{B}_2$ (e.g. for $E = \prod_u E_i$ with $E_i$ a uniformly random graph on $V_i$ we have $E \in \mathcal{B}_2 \setminus (\mathcal{B}_1 \times \mathcal{B}_1)$).
Szemerédi’s regularity lemma as a measure-theoretic statement: Elek-Szegedy, Tao, Towsner, ...

- [Szemerédi’s regularity lemma] Given $E \in \mathcal{B}_2$ and $\varepsilon > 0$, there is a decomposition of the form

$$1_E = f_{\text{str}} + f_{\text{qr}} + f_{\text{err}},$$

where:

- $f_{\text{str}} = \sum_{i \leq n} d_i 1_{A_i}(x) 1_{B_i}(y)$ for some $n \in \mathbb{N}$, $A_i, B_i \in \mathcal{B}_1$ and $d_i \in [0, 1]$ (so $f_{\text{str}}$ is $\mathcal{B}_1 \times \mathcal{B}_1$-simple),

- $f_{\text{err}} : V^2 \to [-1, 1]$ and $\int_{V^2} |f_{\text{err}}|^2 \, d\mu^2 < \varepsilon$,

- $f_{\text{qr}}$ is quasi-random: for any $A, B \in \mathcal{B}_1$ we have

$$\int_{V^2} 1_A(x) 1_B(y) f_{\text{qr}}(x, y) \, d\mu^2 = 0.$$

- Under what conditions on $E$ can the quasi-random part be omitted?
VC-dimension

- Given $E \subseteq V^2$ and $x \in V$, let $E_x = \{y \in V : (x, y) \in E\}$ be the $x$-fiber of $E$.
- A graph $E \subseteq V^2$ has VC-dimension $\geq d$ if there are some $y_1, \ldots, y_d \in V$ such that, for every $S \subseteq \{y_1, \ldots, y_d\}$ there is $x \in V$ so that $E_x \cap \{y_1, \ldots, y_d\} = S$.
- **Example.** If $E_i$ is a random graph on $V_i$ and $(V, E) = \prod_{i \in \mathcal{U}} (V_i, E_i)$, then $\text{VC}(E) = \infty$.
- **Example.** If $E$ is definable in an NIP theory (e.g. $E$ is semialgebraic), then $\text{VC}(E) < \infty$. 
Regularity lemma for graphs of finite VC-dimension

- [Alon, Fischer, Newman] [Lovasz, Szegedy] [Hrushovski, Pillay, Simon], [C., Starchenko] If $E \in \mathcal{B}_2$ and $\text{VC}(E) < \infty$, then:
  - $E \in \mathcal{B}_1 \times \mathcal{B}_1$,
  - the number of rectangles needed to approximate $E$ within $\varepsilon$ is bounded by a polynomial in $\frac{1}{\varepsilon}$. 
We discuss 3-hypergraphs for simplicity.

We have $B_3 \supseteq B_1 \times B_1 \times B_1, B_2 \times B_1$, etc.

Moreover, let $B_{3,2} \subseteq B_3$ be the $\sigma$-algebra generated by intersections of “cylindrical” sets of the form

$$\{(x, y, z) \in V^3 : (x, y) \in A \land (x, z) \in B \land (y, z) \in C\}$$

for some $A, B, C \in B_2$. Again, $B_{3,2} \subsetneq B_3$.

Hypergraph regularity lemma] Any $E \in B_3$ can be decomposed as

$$1_E \approx f(x, y, z) + \sum_{i \leq m} \alpha_i 1_{A_i}(x, y) 1_{B_i}(x, z) 1_{C_i}(y, z) + \sum_{j \leq n} \beta_i 1_{D_i}(x) 1_{F_i}(y) 1_{G_i}(z),$$

where $f$ quasi-random w.r.t. $B_{3,2}$, and $A_i, B_i, C_i \in B_2$ are quasi-random w.r.t $B_1 \times B_1$, and $D_i, F_i, G_i \in B_1$.

Apart from $f$, the rest is $B_{3,2}$-measurable. Under what conditions $E$ is “binary”, i.e. the ternary quasi-random $f$ can be omitted?

[C., Townser] Iff VC$_2$-dimension is finite.
Hypergraph regularity for hypergraphs of slice-wise finite VC-dimension

- Today we discuss the most restrictive case of measurability for hypergraphs with respect to unary sets:
- Let $\mathcal{B}_{3,1} \subseteq \mathcal{B}_3$ be the $\sigma$-algebra generated by intersections of “cylindrical” sets of the form
  $$\{(x, y, z) \in V^3 : x \in A \land y \in B \land z \in C\}$$
  for some $A, B, C \in \mathcal{B}_1$. Note: $\mathcal{B}_{3,1} \subsetneq \mathcal{B}_{3,2}$.
- $E \in \mathcal{B}_3$ has slice-wise finite VC-dimension if for (almost) every $b \in V$, the (binary) fiber
  $E_b = \{(x, y) \in V^2 : (x, y, b) \in E\} \in \mathcal{B}_2$ has finite VC-dimension (and the same for any permutation of the variables).
- $[\text{C., Starchenko}] + [\text{C., Townser}]$ $E \in \mathcal{B}_3$ is slice-wise finite VC-dimension iff $E \in \mathcal{B}_{3,1}$. 
Stability and $\mu$-stability

- Fix $E \in \mathcal{B}_2$.
- A ladder for $E$ of height $d$ is a tuple $\bar{a} \sim \bar{b} = (a_i : i \in d) \sim (b_i : i \in d)$ with $a_i \in V$, $b_i \in V$ such that for every $i, j \in d$ we have $(a_i, b_j) \in E \iff i \leq j$.
- $E$ is $d$-stable if there are no ladders of height $d$ for $E$, and stable if it is ladder $d$-stable for some $d \in \omega$.
- For regularity lemmas, we can ignore measure 0 ladders, so it is natural to relax the definition as follows:
- A $\mu$-ladder for $E$ of height $d$ is a tuple $\bar{b} = (b_j : j \in d)$ so that for every $i \in d$ we have $\mu \left( \bigcap_{i \leq j} E_{b_j} \setminus \left( \bigcup_{j > i} E_{b_j} \right) \right) > 0$.
- For $E \in \mathcal{B}_2$, let $\text{Lad}^{\mu, E, d} \in \mathcal{B}_d$ be the set of all $\bar{b} = (b_i : i \in d)$ so that $\bar{b}$ is a $\mu$-ladder for $E$ of height $d$.
- $E \in \mathcal{B}_2$ is $d$-$\mu$-stable if $\mu \left( \text{Lad}^{\mu, E, d} \right) = 0$. And $E$ is $\mu$-stable if it is ladder $d$-$\mu$-stable for some $d \in \omega$. 
Regularity for $\mu$-stable graphs and hypergraphs

- A set $A \in \mathcal{B}_1$ is perfect for $E \in \mathcal{B}_2$ if
  \[ \mu(\{b \in V : \mu(E_b \cap A) > 0 \land \mu(A \setminus E_b) > 0\}) = 0. \]

- Note: if $A, B \in \mathcal{B}_1$ are perfect for $E$, then
  \[ \frac{\mu(E \cap (A \times B))}{\mu(A \times B)} \in \{0, 1\}. \]

- A simplified version of [Malliaris-Shelah]: Assume that $E \in \mathcal{B}_2$ is $\mu$-stable. Then there exist countable partitions $V = \bigsqcup_{i \in \omega} A_i$ and $V = \bigsqcup_{j \in \omega} B_j$ into perfect sets. In particular, for each $i, j \in \omega$, \[ \frac{\mu(E \cap (A_i \times B_j))}{\mu(A_i \times B_j)} \in \{0, 1\}. \]

- What about hypergraphs?

- We say that $E \in \mathcal{B}_3$ is (partition-wise) $\mu$-stable if the binary relation $E(x;yz)$ is $\mu$-stable, and the same for any other partition of the variables.

- [C., Starchenko], [Ackerman, Freer, Patel] If $E \in \mathcal{B}_3$ is $\mu$-stable, then there exist countable partitions $A_i, B_j, C_k$ of $V$ into perfect sets (for $E$ viewed as a binary relation). In particular, for each $i, j, k \in \omega$, \[ \frac{\mu(E \cap (A_i \times B_j \times C_k))}{\mu(A_i \times B_j \times C_k)} \in \{0, 1\}. \]
Let $\mathcal{H}$ be a family of finite $k$-partite $k$-hypergraphs of the form $H = (E; X_1, \ldots, X_k)$ with $E \subseteq \prod_{i=1}^k X_i$ and $X_i$ finite.

We say that $\mathcal{H}$ satisfies stable regularity if for every $\varepsilon \in \mathbb{R}_{>0}$ there exists some $N = N(\varepsilon)$ such that: for any $H = (E; X_1, \ldots, X_k) \in \mathcal{H}$ and any probability measures $\mu_i$ on $X_i$ there exists $N' \leq N$ and partitions $X_i = \bigsqcup_{0 \leq t < N'} A_{i,t}$ so that for any $0 \leq t_1, \ldots, t_k \leq N'$ we have

$$\frac{\mu \left( E \cap (A_{1,t_1} \times \ldots \times A_{k,t_k}) \right)}{\mu \left( A_{1,t_1} \times \ldots \times A_{k,t_k} \right)} \in [0, \varepsilon) \cup (1 - \varepsilon, 1],$$

where $\mu$ is the product measure of $\mu_1, \ldots, \mu_k$. 

Strong ("meta-stable") stable regularity for families of finite graphs

Let $\mathcal{H}$ be a family of finite $k$-partite $k$-hypergraphs. We say that $\mathcal{H}$ satisfies strong stable regularity if for every $\varepsilon \in \mathbb{R}_{>0}$ and every function $f : \mathbb{N} \to (0, 1)$ there exists some $N = N(f, \varepsilon)$ such that: for any $H = (E; X_1, \ldots, X_k) \in \mathcal{H}$ and any probability measures $\mu_t$ on $X_t$ there exists $N' \leq N$ and partitions $X_i = \bigcup_{0 \leq t < N'} A_{i,t}$ so that:

1. $\mu_i(A_{i,0}) \leq \varepsilon$ for all $1 \leq i \leq k$;
2. for any $1 \leq t_1, \ldots, t_k < N'$ we have
   \[\frac{\mu(E \cap (A_{1,t_1} \times \ldots \times A_{k,t_k})))}{\mu(A_{1,t_1} \times \ldots \times A_{k,t_k})} \in [0, f(N')) \cup (1 - f(N'), 1];\]
3. for each $1 \leq t_1 \leq N'$ we have: for all $(x_2, \ldots, x_k)$ in $A_{2,0} \times X_3 \times \ldots \times X_k$ outside of a subset of measure $\leq f(N')$,
   \[\frac{\mu(E_{(x_2,\ldots,x_k)} \cap A_{1,t_1})}{\mu(A_{1,t_1})} \in [0, f(N')) \cup (1 - f(N'), 1],\]
   and the same for every permutation of the coordinates.
Stable regularity vs strong stable regularity

1. Conditions (1),(2) were considered [Terry, Wolf], [Chavarria, Conant, Pillay].

2. For any arity $k$ hypergraphs, strong stable regularity implies stable regularity.

3. For any $k$ and $\mathcal{H}$ a family of $k$-ary hypergraphs, TFAE:
   - $\mathcal{H}$ satisfies strong stable regularity;
   - in every ultraproduct $H = (E; X_1, \ldots, X_k)$ of $\mathcal{H}$, there exist countable partitions of each $X_i$ into perfect sets from $\mathcal{B}_1$.
   - there is $d \in \mathbb{N}$ so that every $H \in \mathcal{H}$ is partition-wise $d$-stable.

4. For $k = 2$ and $\mathcal{H}$ a family of graphs, everything is equivalent:
   - $\mathcal{H}$ satisfies stable regularity;
   - $\mathcal{H}$ satisfies strong stable regularity;
   - there exist countable perfect partitions in the ultraproduct;
   - there is $d \in \mathbb{N}$ so that every $H \in \mathcal{H}$ is $d$-stable.

5. But not for $k \geq 3$! The relation $E(x, y, z)$ given by $x = y < z$ satisfies stable regularity, but not strong stable regularity (so $E$ is not partition-wise stable).

6. We view the strong version of regularity as the correct and more robust higher arity notion.
Regularity for slice-wise $\mu$-stable hypergraphs

- [Terry-Wolf] Do slice-wise stable $E \in \mathcal{B}_3$ also satisfy stable regularity?
- (This seems to be the last remaining question about measurability with respect to unary sets.)
- We say that $E \in \mathcal{B}_3$ is slicewise $\mu$-stable if the binary fiber $E_b \in \mathcal{B}_2$ is $\mu$-stable for almost all $b \in V$, and the same for every permutation of the coordinates.

Theorem (C., Towsner)

No! But we have the next best thing:
Suppose that $E \in \mathcal{B}_3$ is slice-wise $\mu$-stable. Then there exist countable partitions $A_i, B_j, C_k$ of $V \times V$ so that: each $A_i$ is perfect for the relation $E(xy; z)$, and $A^i = A^i \times A^i$ is a rectangle with $A^i \times A^i, A^i \times A^i \in \mathcal{B}_1$, and same for $B_j, C_k$ with respect to the other partitions of the variables. In particular, for every $i, j$,
$$\frac{\mu(E(x,y,z) \cap A_i(x,y) \cap B_j(x,z))}{\mu(A_i(x,y) \cap B_j(x,z))} \in \{0, 1\}. \ (And \ same \ for \ any \ two \ out \ of \ \{A, B, C\} \ instead \ of \ A, B.)$$
So let $E \in X \times Y \times Z$ be slice-wise $\mu$-stable.

Then for (almost) every $x \in X$, $E_x \subseteq Y \times Z$ is $\mu$-stable, so by the stable graph regularity can decompose $Y, Z$ into perfect sets with respect to $E_x$. But a priori there is no relation between such decompositions of $Y, Z$ for different $x$!

To achieve uniformity, we are going to do a number of repartitions in a “definable” way.

First, a general “symmetrization” result for binary relations:
Lemma

Assume \( A \subseteq X \times Y \) with \( A \in \mathcal{B}_{X \times Y} \). Then there exist countable partitions \( X = \bigsqcup_{i \in \omega} U_i \) with \( U_i \in \mathcal{B}_X \) and \( Y = \bigsqcup_{i \in \omega} V_i \) with \( V_i \in \mathcal{B}_Y \) such that for each \( i \in \omega \) we have:

1. \( \mu \left( (A \cap (U_i \times Y)) \triangle (A \cap (X \times V_i)) \right) = 0 \),

2. for any \( U' \subseteq U_i, U' \in \mathcal{B}_X \) such that both \( \mu \left( A \cap (U' \times Y) \right) > 0 \) and \( \mu \left( A \cap ((U_i \setminus U') \times Y) \right) > 0 \), for any \( V' \subseteq V_i, V' \in \mathcal{B}_Y \) we have
   \( \mu \left( (A \cap (U' \times Y)) \triangle (A \cap (U_i \times V')) \right) > 0 \).

In particular, \( A \) is almost contained in the rectangles on the diagonal, that is \( \mu \left( A \setminus \bigcup_{i \in \omega} (U_i \times V_i) \right) = 0 \).
Getting $\mu$-stable graph regularity uniformly in fibers

As mentioned earlier, we have regularity for hypergraphs of slice-wise finite VC-dimension uniformly over fibers:

Lemma
Assume $E \in B_{X \times Y \times Z}$ is such that for almost all $z \in Z$, the binary relation $E_z \in B_{X \times Y}$ is $\mu$-NIP. Then there exist $P^i \in B^E_{X \times Z}$, $Q^i \in B^E_{Y \times Z}$ for $i \in \omega$ such that for almost every $z \in Z$ we have $\chi_{E_z}(x, y) = \sum_{i \in \omega} \chi_{P^i_z}(x) \cdot \chi_{Q^i_z}(y)$.

After some “definable” refining repartitions using this uniformity and symmetrizations, we obtain uniformity for stable partitions:

Lemma
Suppose that $E \in B_{X \times Y \times Z}$, $E_x \in B_{Y \times Z}$ is $\mu$-stable for almost all $x \in X$. Then there is a partition of $X \times Y$ into countably many sets $A^i \in B_{X \times Y}$, $i \in \omega$, so that for almost every $x \in X$, $(A^i_x : i \in \omega)$ is a partition of $Y$ into countably many sets perfect for $E_x$ (viewed as a binary relation on $(X \times Y) \times Z$).
Partitioning $X \times Y$ into perfect sets

Using this and some more work we obtain a partition of $X \times Y$ into perfect sets:

**Proposition.** Suppose that $E \in \mathcal{B}_{X \times Y \times Z}$, $E_x \in \mathcal{B}_{Y \times Z}$ is $\mu$-stable for almost all $x \in X$, and $E_y \in \mathcal{B}_{X \times Z}$ is $\mu$-stable for almost all $y \in Y$. Then there is a partition of $X \times Y$ into $\mathcal{B}_{X \times Y}^{E}$-measurable sets perfect for $E$, viewed as a binary relation on $(X \times Y) \times Z$.

However, we cannot hope to also partition $Z$ into perfect sets for $E \subseteq (X \times Y) \times Z$, as we did with ordinary stability:

Take $X = Y = Z = [0, 1]$ and let $E := \{(x, y, z) : x = y < z\}$, then $E$ is slicewise stable. Place the Lebesgue measure on $Z$, and place discrete measures on $X$ and $Y$ which place a positive measure on each rational number in $[0, 1]$. Now if $A \subseteq Z$ has positive Lebesgue measure, we can always choose $q \in \mathbb{Q} \cap [0, 1]$ so that both $A \cap [0, q)$ and $A \cap (q, 1]$ have positive measure, that is $0 < \mu (E_{(q,q)} \cap A) < \mu (A)$. But $\mu (\{(q, q)\}) > 0$, so the set $A$ is not perfect.
One direction of stability and the opposite slicewise stability

In this special case the results we have suffice to give a positive answer to the question of Terry and Wolf.

**Theorem**

Assume that $E \in \mathcal{B}_{X \times Y \times Z}$ is $\mu$-stable viewed as a binary relation between $X \times Y$ and $Z$, and the slices $E_z \in \mathcal{B}_{X \times Y}$ are $\mu$-stable for almost all $z \in Z$. Then for every $\varepsilon > 0$ there exist finite partitions $X = \bigsqcup_{i \in I} X_i$, $Y = \bigsqcup_{j \in J} Y_j$, $Z = \bigsqcup_{k \in K} Z_k$ with $X_i \in \mathcal{B}_X$, $Y_j \in \mathcal{B}_Y$, $Z_k \in \mathcal{B}_Z$ so that for every $(i, j, k) \in I \times J \times K$ we have

$$\frac{\mu(E \cap (X_i \times Y_j \times Z_k))}{\mu(X_i \times Y_j \times Z_k)} \in [0, \varepsilon) \cup (1 - \varepsilon, 1].$$
But we only have slice-wise stability in all three directions! Some analysis of infinite (infinitely branching) trees of partitions, with infinite branches tackled by $\mu$-stability on various repartitions of coordinates and slices, allows us to get:

**Proposition.** Suppose that $E \in \mathcal{B}_{X \times Y \times Z}$, the slices $E_x \in \mathcal{B}_{Y \times Z}$ are $\mu$-stable for almost all $x \in X$, and the slices $E_y \in \mathcal{B}_{X \times Z}$ are $\mu$-stable for almost all $y \in Y$. Then there exist a countable partition $X \times Y = \bigcup_{i \in \omega} A^i$ with each $A^i \in \mathcal{B}_{X \times Y}$ perfect for the relation $E \subseteq (X \times Y) \times Z$, and a countable partition $Y \times Z = \bigcup_{j \in \omega} B^j$ into rectangles $B^j = B^j_Y \times B^j_Z$ for some $B^j_Y \in \mathcal{B}_Y$, $B^j_Z \in \mathcal{B}_Z$, so that for each $i, j \in \omega$, either $A^i \cap B^j \subseteq^0 E$ or $(A^i \cap B^j) \cap E =^0 \emptyset$. 


Finally...

Finally, combining all of the above and some more repartitions, we obtain:

**Proposition.** Suppose that $E \in \mathcal{B}_{X \times Y \times Z}$ is slicewise $\mu$-stable. Then there exist a countable partition $X \times Y = \bigcup_{i \in \omega} A^i$ so that each $A^i$ is perfect for the relation $E \subseteq (X \times Y) \times Z$, and $A^i = A^i, X \times A^i, Y$ is a rectangle with $A^i, X \in \mathcal{B}_X$, $A^i, Y \in \mathcal{B}_Y$.

From which the main theorem quickly follows!

A slicewise stable counterexample to stable hypergraph regularity: Let $X := \{0, 1, 2\}^\omega$, and $(x, y, z) \in E$ holds if, for the first $n$ such that $|x(n), y(n), z(n)| > 1$, $|x(m), y(n), z(n)| = 3$. (At the first coordinate where they are not all the same, they are all different.)
Thank you!

- “Hypergraph regularity and higher arity VC-dimension” with Henry Towsner, arXiv:2010.00726
- “A regularity lemma for slice-wise stable hypergraphs”, with Henry Towsner, in preparation