Fields and model-theoretic classification, 1

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Definable sets

- Let $\mathcal{M} = (M, R_i, f_i, c_i)$ denote a first-order structure with some distinguished relations $R_i \subseteq M^{k_i}$, functions $f_i : M^{k_i} \to M$ and constants $c_i \in M$. Here $\mathcal{L} = (R_i, f_i, c_i)$ is the *language* of $\mathcal{M}$.
- For example, a group is naturally viewed as a structure $\left(G, \cdot, ^{-1}, 1\right)$, as well as any ring $(R, +, \cdot, 0, 1)$, ordered set $(X, <)$, graph (X, E), etc.
- A (partitioned) first-order formula $\phi(x, y)$ is an expression of the form $\forall z_1 \exists z_2 \ldots \forall z_{2n-1} \exists z_{2n} \psi(x, y, \bar{z})$, where ψ is a Boolean combination of the (superpositions of) basic relations and functions, and x, y are tuples of variables.
- Given some parameters $b \in M^{|y|}$, $\phi(x, b)$ is *an instance* of ϕ and defines a set $\phi(M, b) = \left\{a \in M^{|x|} : \mathcal{M} \models \phi(a, b)\right\}$.
- Subsets of M^n of this form are called *definable* and form a Boolean algebra.
- E.g. in a group G, the set of solutions of a formula $\phi(x) = \forall y \, (x \cdot y = y \cdot x)$ is the center of G.

Complete theories

- If formula with no free variables is called a *sentence*, and it is either true or false in $\mathcal{M}$.
- The theory of $\mathcal{M}$, or $\mathrm{Th}(M)$, is the collection of *all* sentences that are true in $\mathcal{M}$.
- Two $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ are *elementarily equivalent* if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.
- If $\mathcal{M} \subseteq \mathcal{N}$ and for every formula $\phi(x) \in \mathcal{L}$ and $a \in M^{|x|}$, $\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(a)$, then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, denoted $\mathcal{M} \preceq \mathcal{N}$.
- In first approximation, model theory studies *complete* theories T (equivalently, structures up to *elementary equivalence*) and their corresponding categories of definable sets.
- In second approximation, up to bi-interpretability.

Gödelian phenomena

- Consider $(\mathbb{N}, +, \times, 0, 1)$. The more quantifiers we allow, the more complicated sets we can define (e.g. non-computable sets, and in fact a large part of mathematics can be encoded — "Gödelian phenomena").
- Similarly, by a result of Julia Robinson, the field $(\mathbb{Q}, +, \times, 0, 1)$ interprets $(\mathbb{N}, +, \times, 0, 1)$, so it is as complicated.
- In particular, no hope of describing the structure of definable sets in any kind of "geometric" manner.
- On the other hand, definable sets in $(\mathbb{C}, +, \times, 0, 1)$ are within the scope of algebraic geometry, and admit a beautiful and elaborate theory (see later).
- Hence, the Boolean algebra of definable sets is "wild" in the first case, and "tame" in the second.
- How to make this distinction between wild and tame structures precise and independent of the specific language in which these structures are considered?

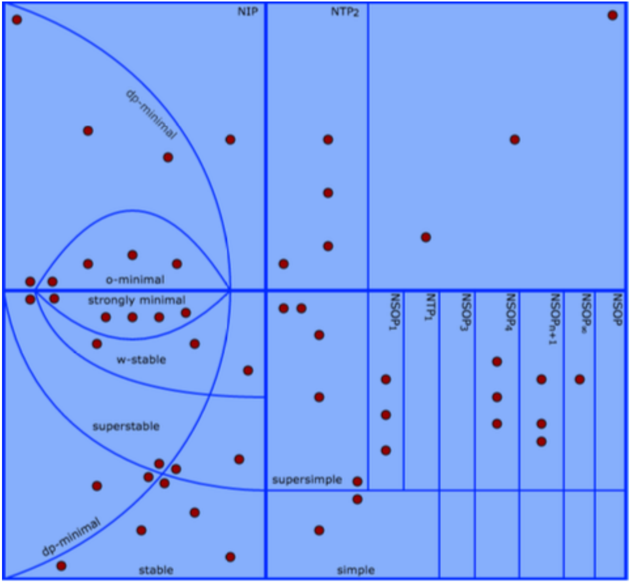
Model theoretic classification

- [Morley, 1965] Let T be a countable first-order theory. Assume T has a unique model (up to isomorphism) of size κ for some uncountable cardinal κ. Then **for any** uncountable cardinal λ it has a unique model of size λ.
- Morley's conjecture: for any T, the function

$$f_T : \kappa \mapsto |\{M : M \models T, |M| = \kappa\}|$$

  is non-decreasing on uncountable cardinals.
- Shelah's "dividing lines" solution: describe all possible functions, distinguished by T being able to encode certain explicit combinatorial configurations in a definable manner. If it does, demonstrate that there are as many models as possible, if it doesn't, develop some dimension theory to describe its models.
- Later, Zilber, Hrushovski and others — geometric stability theory. In order to understand *arbitrary* theories, it is essential to understand groups and fields definable in them.

(Partial) Classification picture



http://www.forkinganddividing.com/

Model theoretic classification of groups and fields

- Hence, understanding tame groups and fields not only provides important examples, but is also essential for the general theory.
- Classifying groups is as complicated as classifying arbitrary theories:
- [Mekler, 81] For every theory T in a finite relational language, there is a theory T' of pure groups (nilpotent, class 2) which interprets T and is in the same tameness class as T, e.g. T' is stable/simple/NIP/NTP$_2$, assuming T was (T' is not interpretable in T in general).
- Remarkably, for fields, model-theoretic dividing lines tend to coincide with natural algebraic properties.

Types

- Let T be fixed, $\mathcal{M} \models T$.
- A *partial type* $\pi(x)$ over a set of parameters $A \subseteq M$ is a collection of formulas over A such that for any finite $\pi_0 \subseteq \pi$, there is some $a \in M^{|x|}$ such that $a \models \pi_0(x)$.
- $\mathcal{M}$ is κ-saturated if every n-type over every $A \subseteq M, |A| < \kappa$ is realized in M.
- (Compactness theorem) Every $\mathcal{M}$ admits a κ-saturated elementary extension $\mathcal{N} \succeq \mathcal{M}$, for any κ.
- Let $\mathcal{M} = (\mathbb{R}, +, \times, <, 0, 1)$, and consider $\pi(x) = \left\{0 < x < \frac{1}{n} : n \in \mathbb{N}\right\}$. Not realized in $\mathbb{R}$ (thus $\mathbb{R}$ is not $\aleph_0$-saturated). Passing to some $\aleph_0$-saturated $\mathbb{R}^* \succ \mathbb{R}$, the set of solutions of $\pi(x)$ in $\mathbb{R}^*$ is the set of "infinitesimal" elements, and one can do *non-standard analysis* working in $\mathbb{R}^*$.
- A complete type $p(x)$ over A is a maximal (under inclusion) partial type over A (equivalently, an ultrafilter in the Boolean algebra of A-definable subsets of $M^{|x|}$). Let $S_x(A)$ denotes the space of all complete types over A (*Stone dual)*.

Stability

- Given a theory T in a language $\mathcal{L}$, a (partitioned) formula $\phi(x, y) \in \mathcal{L}$ (x, y are tuples of variables), a model $\mathcal{M} \models T$ and $b \in M^{|y|}$, let $\phi(M, b) = \{a \in M^{|x|} : M \models \phi(a, b)\}$.
- Let $\mathcal{F}_{\phi,M} = \{\phi(M, b) : b \in M^{|y|}\}$ be the family of ϕ-definable subsets of M. Dividing lines can be typically expressed as certain conditions on the combinatorial complexity of the families $\mathcal{F}_{\phi,M}$ (independent of the choice of $\mathcal{M}$).

Definition

1. A (partitioned) formula $\phi(x, y)$ is stable if **there are no** $\mathcal{M} \models T$ and $(a_i, b_i : i < \omega)$ with $a_i \in M^{|x|}, b_i \in M^{|y|}$ such that
$$\mathcal{M} \models \phi(a_i, b_j) \iff i \leq j.$$
2. A theory T is stable if it implies that all formulas are stable.

- E.g. $(\mathbb{Q}, <)$ is not stable.

Stability is equivalent to few types

Definition

T is *κ-stable* if $\sup \{|S_1(M)| : \mathcal{M} \models T, |\mathcal{M}| = \kappa\} \leq \kappa$ (i.e. the space of types is as small as possible).

Fact

Let T be a complete theory. TFAE:

1. *T is stable.*
2. *T is κ-stable for some κ.*
3. *T is κ-stable for every κ with $\kappa = \kappa^{|T|}$.*

- It is easy to see that if T is κ-stable, then the same bound holds for $S_n(M)$ *for any* $n \in \omega$. Hence it is enough to check that all formulas $\phi(x, y)$ with $|x| = 1$ are stable.

Examples of stable fields: algebraically closed fields

- We consider $\mathrm{Th}(\mathbb{C}, +, \times, 0, 1)$.
- Recall: a field K is *algebraically closed* if it contains a root for every non-constant polynomial in $K[x]$ (equivalently, no proper algebraic extensions).
- By the fundamental theorem of algebra, $\mathbb{C}$ is algebraically closed (and this condition is expressible as an infinite collection of first-order sentences).
- For $p = 0$ or prime, let ACF_p be the theory of algebraically closed fields of characteristic p.
- [Tarski] ACF_p is a complete theory eliminating quantifiers.

Examples of stable fields: algebraically closed fields

- In particular, if $\mathcal{M} \models \mathrm{ACF}_p$, then every subset of M is either finite or cofinite. Theories with this property are called *strongly minimal*.
- If T is strongly minimal, then it is ω-stable.
  The complete 1-types over $\mathcal{M} \models T$ are of the form $x = a$ for some $a \in M$, plus one non-algebraic type (axiomatized by $\{x \neq a : a \in M\}$), hence $|S_1(\mathcal{M})| \leq |M|$.

Examples of stable fields: separably closed fields

- For a field K, we let K^{alg} denote its algebraic closure (i.e. an algebraic extension of K which is algebraically closed, unique up to an isomorphism fixing K pointwise).

Definition

A field K is *separably closed* if every polynomial $P(X) \in K[X]$ whose roots in K^{alg} are distinct, has at least one root in K. (Equivalently, every irreducible polynomial over K is of the form $X^{p^k} - a$, where p is the characteristic)

- Any separably closed field of char 0 is algebraically closed.
- If $\mathrm{char}\,(K) = p$, then K^p is a subfield. If the degree of $[K : K^p]$ is finite, it is of the form p^e, and e is called the *degree of imperfection* of K. For any $e \in \mathbb{N}$, let $\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of char p with the degree of imperfection e, and let $\mathrm{SCF}_{p,\infty}$ be the theory of separably closed fields of char p with infinite degree of imperfection.

Examples of stable fields: separably closed fields

- These are all complete theories of separably closed fields, and they eliminate quantifiers after naming a basis and adding some function symbols to the language.
- [Wood, 79] All these theories are stable (and in the non-algebraically closed case, strictly stable, i.e. not superstable).
- Separably closed fields played a key role in Hrushovski's proof of the Geometric Mordell Lang conjecture in positive characteristic.

Other stable fields?

- [Macintyre, 71] All infinite ω-stable fields are algebraically closed.
- [Cherlin, Shelah, 80] All infinite superstable fields are algebraically closed.
- **Open problem:** are all infinite stable fields separably closed?
- Little progress has been made so far. A noteworthy result (will be discussed later):
- [Scanlon, 91] If K is an infinite stable field of characteristic p, then K has no finite Galois extensions of degree divisible by p.
- We sketch a proof of Macintyre's theorem, key ingredients:
  - chain condition for definable groups,
  - theory of group generics,
  - some Galois theory.

Morley rank in ω-stable theories

- If T is ω-stable, then (working in a saturated model $\mathbb{M}$) to every definable set we can inductively assign an ordinal-valued rank, *Morley rank*, by:
- $\mathrm{RM}(X) = 0$ iff X is finite and $\mathrm{RM}(X) \geq \alpha + 1$ if and only if there are pairwise disjoint definable subsets $\{Y_i : i \in \omega\}$ of X with $\mathrm{RM}(Y_i) \geq \alpha$ for all $i \in \omega$.
  (otherwise can build a tree of dividing formulas which would produce too many types).
- Given a type $p \in S_x(A)$,
  $\mathrm{RM}(p) = \inf \{\mathrm{RM}(\phi(x)) : \phi(x) \in p\}$.
- Has many "dimension-like" properties, in particular is preserved by definable bijections.
- Now if $H \leq G$ are definable in an ω-stable theory and $[G : H]$ is infinite, then $\mathrm{RM}(H) < \mathrm{RM}(G)$ (we can take Y_i above to be the infinitely many cosets of H in G).
- As there are no infinite decreasing chains of ordinals and G has a Morley rank, one obtains:

Chain conditions and connected components in ω-stable groups

Fact (Descending Chain Condition, DCC). If G is a group definable in an ω-stable theory, then there is no infinite descending chain of definable subgroups $G > G_1 > G_2 > \dots$.

Corollary. If G is an ω-stable group and $\{H_i : i \in I\}$ is a collection of definable subgroups, then there is some *finite* $I_0 \subseteq I$ such that

$$\bigcap_{i \in I} H_i = \bigcap_{i \in I_0} H_i.$$

Corollary. If G is an ω-stable group, then it has a *connected component* $G^0 \leq G$ — the smallest definable finite index subgroup of G. Moreover:

- G^0 is a normal subgroup of G and is definable over $\emptyset$.
- If $\sigma : G \to G$ is a definable group automorphism, then σ fixed G^0 setwise.

Generics in ω-stable groups

- Let G be a definable group (in a saturated structure $\mathbb{M}$).
- A definable set $X \subseteq G$ is called *(left-)generic* if G can be covered by finitely many translates of X.
- A type $p \in S_G(M)$ over a small model M is generic if it only contains generic formulas.
- $\iff \mathrm{RM}(p) = \mathrm{RM}(G) \iff \mathrm{Stab}(p) = G^0_M$.
- We say that $a \in \mathbb{M}$ is generic over K if $\mathrm{RM}(\mathrm{tp}(a/M)) = \mathrm{RM}(G)$.
- **Fact.** G has a unique generic type if and only if G is *connected*, i.e. $G = G^0$.
- This generalizes the notion of a "generic point" of an algebraic group.

ω-stable fields are algebraically closed, 1

1. Let $(K, +, \cdot, \ldots)$ be an infinite ω-stable field, w.l.o.g. K is saturated.
2. The additive group $(K, +, \ldots)$ is connected, i.e. $K^0 = K$. For $a \in K \setminus \{0\}$, $x \mapsto ax$ is a definable group automorphism — must fix K^0 — hence $aK^0 = K^0$, so K^0 is an ideal of K. Because K is a field, there are no proper ideals.
3. As K is connected as an additive group, there is a unique type of max Morley rank, thus the mult. group $(K^\times, \cdot, \ldots)$ is also connected.
4. For each $n \in \omega$, the map $x \mapsto x^n$ is a mult. homomorphism. If a is generic, then a^n is also generic (interalgebraic with a).
5. Thus K^n contains the generic, and as the mult. group is connected, $K^n = K$ and every element has an nth root.
6. In particular, if $\operatorname{char}(K) = p > 0$, then every element has pth root, thus K is perfect.
7. Suppose $\operatorname{char}(K) = p > 0$. The map $x \mapsto x^p + x$ is an additive homomorphism. If a is generic, then $a^p + a$ is also generic, and as above the map is surjective.

ω-stable fields are algebraically closed, 2

Claim 1. Suppose K is an infinite ω-stable field containing all mth roots of unity for $m \leq n$. Then K has no proper Galois extensions of degree n.

- Let L/K be a counterexample with the least possible n, let q be a prime dividing n.
- By Galois theory, there is $K \subseteq F \subset L$ such that L/F is Galois of degree q.
- The field F is a finite algebraic extension of K, hence interpretable in K, hence F is ω-stable.
- By minimality of n, $F = K$ and $n = q$.
- If $\operatorname{char}(K) = 0$, by Galois theory the minimal polynomial of L/K is $X^q - a$ for some $a \in K$. But every element of K has a qth root, thus $X^q - a$ is reducible, a contradiction.
- If $\operatorname{char}(K) = p = q$, by Galois theory the minimal polynomial of L/K is $X^p + X - a$ for some $a \in K$. But the map $x \mapsto x^p + x - a$ is surjective, thus $X^p + X - a$ is reducible, a contradiction.

ω-stable fields are algebraically closed, 3

Claim 2. If K is an infinite ω-stable field, then K contains all roots of unity.
Let n be the least such that K doesn't contain all nth roots of unity. Let ξ be a primitive nth root of unity. Then $K(\xi)$ is a Galois extension of K of degree at most $n-1$. This contradicts the previous claim.

- Because K contains all roots of unity, the first claim implies that K has no proper Galois extensions. Because K is perfect, K is algebraically closed.