VC-dimension in model theory and other subjects

Artem Chernikov

(Paris 7 / MSRI, Berkeley)

UCLA,

2 May 2014

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- Let $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subset S, |B| = n\}$.

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- Let $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subset S, |B| = n\}$.
- If the VC dimension of $\mathcal{F}$ is infinite, then $\pi_{\mathcal{F}}(n) = 2^n$ for all n. However,

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- Let $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subset S, |B| = n\}$.
- If the VC dimension of $\mathcal{F}$ is infinite, then $\pi_{\mathcal{F}}(n) = 2^n$ for all n. However,

Fact

[Sauer-Shelah lemma] If $\mathcal{F}$ has VC dimension $\leq d$, then for $n \geq d$ we have $\pi_{\mathcal{F}}(n) \leq \sum_{i \leq d} \binom{n}{i} = O(n^d)$.

VC-dimension

- Let $\mathcal{F}$ be a family of subsets of a set X.
- For a set $B \subseteq X$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$.
- We say that $B \subseteq X$ is *shattered* by $\mathcal{F}$ if $\mathcal{F} \cap B = 2^B$.
- Let the *Vapnik-Chervonenkis dimension* (*VC dimension*) of $\mathcal{F}$ be the largest integer n such that some subset of S of size n is shattered by $\mathcal{F}$ (otherwise ∞).
- Let $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subset S, |B| = n\}$.
- If the VC dimension of $\mathcal{F}$ is infinite, then $\pi_{\mathcal{F}}(n) = 2^n$ for all n. However,

Fact

[Sauer-Shelah lemma] If $\mathcal{F}$ has VC dimension $\leq d$, then for $n \geq d$ we have $\pi_{\mathcal{F}}(n) \leq \sum_{i \leq d} \binom{n}{i} = O(n^d)$.

- The bound is tight: consider all subsets of $\{1, \ldots, n\}$ of cardinality less that d.

VC-dimension

- Computational learning theory (PAC),
- computational geometry,
- functional analysis (Bourgain-Fremlin-Talagrand theory),
- model theory (NIP),
- abstract topological dynamics (tame dynamical systems), ...

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F} =$ half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F} =$ half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.
- The class of families of finite VC-dimension is closed under boolean combinations.

Some examples

- $X = \mathbb{R}$, $\mathcal{F}$ = all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F}$ = all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F}$ = half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.
- The class of families of finite VC-dimension is closed under boolean combinations.

- $X = \mathbb{R}^2$, $\mathcal{F}$ = all convex n-gons. Then $VC(\mathcal{F}) = 2n + 1$.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F} =$ half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.
- The class of families of finite VC-dimension is closed under boolean combinations.

- $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex n-gons. Then $VC(\mathcal{F}) = 2n + 1$.
- But: $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex polygons. Then $VC(\mathcal{F}) = \infty$.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F} =$ half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.
- The class of families of finite VC-dimension is closed under boolean combinations.

- $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex n-gons. Then $VC(\mathcal{F}) = 2n + 1$.
- But: $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex polygons. Then $VC(\mathcal{F}) = \infty$.
- $X = \mathbb{R}$, $\mathcal{F} =$ semialgebraic sets of bounded complexity. Then $VC(\mathcal{F})$ is finite.

Some examples

- $X = \mathbb{R}$, $\mathcal{F} =$ all unbounded intervals. Then $VC(\mathcal{F}) = 2$.
- $X = \mathbb{R}^2$, $\mathcal{F} =$ all half-spaces. Then $VC(\mathcal{F}) = 3$.
- $X = \mathbb{R}^d$, $\mathcal{F} =$ half-spaces in $\mathbb{R}^d$. Then $VC(\mathcal{F}) = d + 1$.
- The class of families of finite VC-dimension is closed under boolean combinations.

- $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex n-gons. Then $VC(\mathcal{F}) = 2n + 1$.
- But: $X = \mathbb{R}^2$, $\mathcal{F} =$ all convex polygons. Then $VC(\mathcal{F}) = \infty$.
- $X = \mathbb{R}$, $\mathcal{F} =$ semialgebraic sets of bounded complexity. Then $VC(\mathcal{F})$ is finite.
- Model theory gives a lot of new and more general examples from outside of combinatorial real geometry (a bit later).

The law of large numbers

- Let (X, μ) be a probability space.
- Given a set $S \subseteq X$ and $x_1, \ldots, x_n \in X$, we define $\mathrm{Av}\,(x_1, \ldots, x_n; S) = \frac{1}{n} |S \cap \{x_1, \ldots, x_n\}|$.
- For $n \in \omega$, let μ^n be the product measure on X^n.

The law of large numbers

- Let (X, μ) be a probability space.
- Given a set $S \subseteq X$ and $x_1, \ldots, x_n \in X$, we define $\mathrm{Av}(x_1, \ldots, x_n; S) = \frac{1}{n} |S \cap \{x_1, \ldots, x_n\}|$.
- For $n \in \omega$, let μ^n be the product measure on X^n.

Fact

(Weak law of large numbers) Let $S \subseteq X$ be measurable and fix $\varepsilon > 0$. Then for any $n \in \omega$ we have:

$$\mu^n(\bar{x} \in X^n : |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2} \to 0 \text{ when } n \to \infty.$$

- (i.e., with high probability, sampling on a tuple $(x_1, \ldots, x_n)$ selected at random gives a good estimate of the measure of S.)

VC-theorem

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a family of subsets of X of finite VC-dimension such that:

1. *Every $S \in \mathcal{F}$ is measurable;*

VC-theorem

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a family of subsets of X of finite VC-dimension such that:

1. *Every $S \in \mathcal{F}$ is measurable;*
2. *for each n, the function $f_n(x_1, \ldots, x_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)|$ is a measurable function from X^n to $\mathbb{R}$;*

VC-theorem

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a family of subsets of X of finite VC-dimension such that:

1. *Every $S \in \mathcal{F}$ is measurable;*
2. *for each n, the function $f_n(x_1, \ldots, x_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)|$ is a measurable function from X^n to $\mathbb{R}$;*
3. *for each n, the function $g_n(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mathrm{Av}(x'_1, \ldots, x'_n; S)|$ from X^{2n} to $\mathbb{R}$ is measurable.*

VC-theorem

Fact

[VC theorem] Let (X, μ) be a probability space, and let $\mathcal{F}$ be a family of subsets of X of finite VC-dimension such that:

1. *Every $S \in \mathcal{F}$ is measurable;*
2. *for each n, the function $f_n(x_1, \ldots, x_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)|$ is a measurable function from X^n to $\mathbb{R}$;*
3. *for each n, the function $g_n(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = \sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mathrm{Av}(x'_1, \ldots, x'_n; S)|$ from X^{2n} to $\mathbb{R}$ is measurable.*

Then for every $\varepsilon > 0$ and $n \in \omega$ we have:

$$\mu^n\left(\sup_{S \in \mathcal{F}} |\mathrm{Av}(x_1, \ldots, x_n; S) - \mu(S)| > \varepsilon\right) \leq 8\pi_{\mathcal{F}}(n) \exp\left(-\frac{n\varepsilon^2}{32}\right).$$

VC-theorem and ε-nets

- $\longrightarrow 0$ when $n \longrightarrow \infty$ (as $\pi_{\mathcal{F}}(n)$ is polynomially bounded by Sauer-Shelah).

VC-theorem and ε-nets

- $\longrightarrow 0$ when $n \longrightarrow \infty$ (as $\pi_{\mathcal{F}}(n)$ is polynomially bounded by Sauer-Shelah).
- Of course (1),(2) and (3) hold for any family of subsets of a finite set X. Also if $\mathcal{F}$ is countable then (1) implies (2) and (3).

VC-theorem and ε-nets

- $\longrightarrow 0$ when $n \longrightarrow \infty$ (as $\pi_{\mathcal{F}}(n)$ is polynomially bounded by Sauer-Shelah).
- Of course (1),(2) and (3) hold for any family of subsets of a finite set X. Also if $\mathcal{F}$ is countable then (1) implies (2) and (3).
- Consider $X = \omega_1$, let B be the σ-algebra generated by the intervals, and define $\mu(A) = 1$ if A contains an end segment of X and 0 otherwise. Take $\mathcal{F}$ to be the family of intervals of X. Then $VC(\mathcal{F}) = 2$ but the VC-theorem does not hold for $\mathcal{F}$.

VC-theorem and ε-nets

- $\longrightarrow 0$ when $n \longrightarrow \infty$ (as $\pi_{\mathcal{F}}(n)$ is polynomially bounded by Sauer-Shelah).
- Of course (1),(2) and (3) hold for any family of subsets of a finite set X. Also if $\mathcal{F}$ is countable then (1) implies (2) and (3).
- Consider $X = \omega_1$, let B be the σ-algebra generated by the intervals, and define $\mu(A) = 1$ if A contains an end segment of X and 0 otherwise. Take $\mathcal{F}$ to be the family of intervals of X. Then $VC(\mathcal{F}) = 2$ but the VC-theorem does not hold for $\mathcal{F}$.
- A subset A of X is called an *ε-net* for $\mathcal{F}$ *with respect to* μ *if* $A \cap S \neq \emptyset$ for all $S \in \mathcal{F}$ with $\mu(S) \geq \varepsilon$.

Fact

[ε-nets] If (X, μ) is a probability space and $\mathcal{F}$ is a family of measurable subsets of X with $VC(\mathcal{F}) \leq d$, then for any $r \geq 1$ there is a $\frac{1}{r}$-net for $(X, \mathcal{F})$ with respect to μ of size at most $Cdr \ln r$, where C is an absolute constant.

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\text{fin}}$ denote $\bigcup\{\mathcal{F} \cap B : B$ a finite subset of X with $|B| \geq 2\}$.

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\text{fin}}$ denote $\bigcup\{\mathcal{F} \cap B : B \text{ a finite subset of } X \text{ with } |B| \geq 2\}$.

Definition

$\mathcal{F}$ is said to have a *d-compression scheme* if there is a compression function $\kappa : \mathcal{F}|_{\text{fin}} \to X^d$ and a finite set $\mathcal{R}$ of reconstruction functions $\rho : X^d \to 2^X$ such that for every $f \in \mathcal{F}|_{\text{fin}}$ we have:

1. $\text{range}\,(\kappa\,(f)) \subseteq \text{dom}\,(f)$,

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\mathrm{fin}}$ denote $\bigcup\{\mathcal{F} \cap B : B$ a finite subset of X with $|B| \geq 2\}$.

Definition

$\mathcal{F}$ is said to have a *d-compression scheme* if there is a compression function $\kappa : \mathcal{F}|_{\mathrm{fin}} \to X^d$ and a finite set $\mathcal{R}$ of reconstruction functions $\rho : X^d \to 2^X$ such that for every $f \in \mathcal{F}|_{\mathrm{fin}}$ we have:

1. $\mathrm{range}\left(\kappa\left(f\right)\right) \subseteq \mathrm{dom}\left(f\right)$,
2. $f = \rho\left(\kappa\left(f\right)\right)|_{\mathrm{dom}(f)}$ for at least one $\rho \in \mathcal{R}$.

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\text{fin}}$ denote $\bigcup\{\mathcal{F} \cap B : B$ a finite subset of X with $|B| \geq 2\}$.

Definition

$\mathcal{F}$ is said to have a *d-compression scheme* if there is a compression function $\kappa : \mathcal{F}|_{\text{fin}} \to X^d$ and a finite set $\mathcal{R}$ of reconstruction functions $\rho : X^d \to 2^X$ such that for every $f \in \mathcal{F}|_{\text{fin}}$ we have:

1. $\text{range}\,(\kappa\,(f)) \subseteq \text{dom}\,(f)$,
2. $f = \rho\,(\kappa\,(f))\,|_{\text{dom}(f)}$ for at least one $\rho \in \mathcal{R}$.

- Existence of a compression scheme for $\mathcal{F}$ implies finite VC-dimension.

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\mathrm{fin}}$ denote $\bigcup \{\mathcal{F} \cap B : B$ a finite subset of X with $|B| \geq 2\}$.

Definition

$\mathcal{F}$ is said to have a *d-compression scheme* if there is a compression function $\kappa : \mathcal{F}|_{\mathrm{fin}} \to X^d$ and a finite set $\mathcal{R}$ of reconstruction functions $\rho : X^d \to 2^X$ such that for every $f \in \mathcal{F}|_{\mathrm{fin}}$ we have:

1. $\mathrm{range}(\kappa(f)) \subseteq \mathrm{dom}(f)$,
2. $f = \rho(\kappa(f))|_{\mathrm{dom}(f)}$ for at least one $\rho \in \mathcal{R}$.

- Existence of a compression scheme for $\mathcal{F}$ implies finite VC-dimension.
- **Problem** [Warmuth]. Does every family $\mathcal{F}$ of finite VC-dimension admit a compression scheme? (and if yes, does it admit a $VC(\mathcal{F})$-compression scheme?)

Compression schemes and Warmuth conjecture

- As before, let $\mathcal{F} \subseteq 2^X$ be given. Let $\mathcal{F}|_{\text{fin}}$ denote $\bigcup \{\mathcal{F} \cap B : B \text{ a finite subset of } X \text{ with } |B| \geq 2\}$.

Definition

$\mathcal{F}$ is said to have a *d-compression scheme* if there is a compression function $\kappa : \mathcal{F}|_{\text{fin}} \to X^d$ and a finite set $\mathcal{R}$ of reconstruction functions $\rho : X^d \to 2^X$ such that for every $f \in \mathcal{F}|_{\text{fin}}$ we have:

1. $\text{range}(\kappa(f)) \subseteq \text{dom}(f)$,
2. $f = \rho(\kappa(f))|_{\text{dom}(f)}$ for at least one $\rho \in \mathcal{R}$.

- Existence of a compression scheme for $\mathcal{F}$ implies finite VC-dimension.
- **Problem** [Warmuth]. Does every family $\mathcal{F}$ of finite VC-dimension admit a compression scheme? (and if yes, does it admit a $VC(\mathcal{F})$-compression scheme?)
- Turns out that combining model theory with some more results from combinatorics gives a quite general result towards it.

Model theoretic classification: something completely different?

- Let T be a complete first-order theory in a countable language L. For an infinite cardinal κ, let $I_T(\kappa)$ denote the number of models of T of size κ, up to an isomorphism.
- Note: $1 \leq I_T(\kappa) \leq 2^\kappa$ for all κ.

Model theoretic classification: something completely different?

- Let T be a complete first-order theory in a countable language L. For an infinite cardinal κ, let $I_T(\kappa)$ denote the number of models of T of size κ, up to an isomorphism.
- Note: $1 \leq I_T(\kappa) \leq 2^\kappa$ for all κ.
- Morley's theorem: If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.
- Morley's conjecture: $I_T(\kappa)$ is a non-decreasing function on uncountable cardinals.

Model theoretic classification: something completely different?

- Let T be a complete first-order theory in a countable language L. For an infinite cardinal κ, let $I_T(\kappa)$ denote the number of models of T of size κ, up to an isomorphism.
- Note: $1 \leq I_T(\kappa) \leq 2^\kappa$ for all κ.
- Morley's theorem: If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.
- Morley's conjecture: $I_T(\kappa)$ is a non-decreasing function on uncountable cardinals.
- Shelah's approach: isolate dividing lines, expressed as the ability to encode certain families of graphs in a definable way, such that one can prove existence of many models on the non-structure side of a dividing line and develop some theory on the structure side (forking, weight, prime models, etc). E.g. stability or NIP.

Model theoretic classification: something completely different?

- Let T be a complete first-order theory in a countable language L. For an infinite cardinal κ, let $I_T(\kappa)$ denote the number of models of T of size κ, up to an isomorphism.
- Note: $1 \leq I_T(\kappa) \leq 2^\kappa$ for all κ.
- Morley's theorem: If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.
- Morley's conjecture: $I_T(\kappa)$ is a non-decreasing function on uncountable cardinals.
- Shelah's approach: isolate dividing lines, expressed as the ability to encode certain families of graphs in a definable way, such that one can prove existence of many models on the non-structure side of a dividing line and develop some theory on the structure side (forking, weight, prime models, etc). E.g. stability or NIP.
- Led to a proof of Morley's conjecture. By later work of [Hart, Hrushovski, Laskowski] we know all possible values of $I_T(\kappa)$.

NIP theories

- A formula $\phi(x, y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x, a) \cap M : a \in M\}$ has finite VC-dimension.

NIP theories

- A formula $\phi(x, y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x, a) \cap M : a \in M\}$ has finite VC-dimension.
- Note that this is a property of the theory of M, i.e. if N is elementarily equivalent to M then $\phi(x, y)$ is NIP in N as well.

NIP theories

- A formula $\phi(x, y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x, a) \cap M : a \in M\}$ has finite VC-dimension.
- Note that this is a property of the theory of M, i.e. if N is elementarily equivalent to M then $\phi(x, y)$ is NIP in N as well.
- T is NIP if it implies that every formula $\phi(x, y) \in L$ is NIP.

NIP theories

- A formula $\phi(x, y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x, a) \cap M : a \in M\}$ has finite VC-dimension.
- Note that this is a property of the theory of M, i.e. if N is elementarily equivalent to M then $\phi(x, y)$ is NIP in N as well.
- T is NIP if it implies that every formula $\phi(x, y) \in L$ is NIP.
- **Fact** [Shelah]. If T is not NIP, then it has 2^κ models for any infinite cardinal κ.

NIP theories

- A formula $\phi(x,y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x,a) \cap M : a \in M\}$ has finite VC-dimension.
- Note that this is a property of the theory of M, i.e. if N is elementarily equivalent to M then $\phi(x,y)$ is NIP in N as well.
- T is NIP if it implies that every formula $\phi(x,y) \in L$ is NIP.
- **Fact** [Shelah]. If T is not NIP, then it has 2^κ models for any infinite cardinal κ.

Fact

[Shelah] T is NIP *iff every formula $\phi(x,y)$ with $|x| = 1$ is* NIP*.*

NIP theories

- A formula $\phi(x, y) \in L$ (where x, y are tuples of variables) is NIP in a structure M if the family $\mathcal{F}_\phi = \{\phi(x, a) \cap M : a \in M\}$ has finite VC-dimension.
- Note that this is a property of the theory of M, i.e. if N is elementarily equivalent to M then $\phi(x, y)$ is NIP in N as well.
- T is NIP if it implies that every formula $\phi(x, y) \in L$ is NIP.
- **Fact** [Shelah]. If T is not NIP, then it has 2^κ models for any infinite cardinal κ.

Fact

[Shelah] T is NIP *iff every formula $\phi(x, y)$ with $|x| = 1$ is* NIP*.*

- Curious original proof: holds in some model of ZFC + absoluteness; since then had been finitized using Ramsey theorem.

New examples of VC-families

- Examples of NIP theories:
  - stable theories (e.g. algebraically / separably / differentially closed fields, free groups (Sela), planar graphs),

New examples of VC-families

- Examples of NIP theories:
  - stable theories (e.g. algebraically / separably / differentially closed fields, free groups (Sela), planar graphs),
  - *o*-minimal theories (e.g. real closed fields with exponentiation and analytic functions restricted to $[0, 1]$),

New examples of VC-families

- Examples of NIP theories:
  - stable theories (e.g. algebraically / separably / differentially closed fields, free groups (Sela), planar graphs),
  - *o*-minimal theories (e.g. real closed fields with exponentiation and analytic functions restricted to $[0, 1]$),
  - ordered abelian groups (Gurevich, Schmitt),

New examples of VC-families

- Examples of NIP theories:
  - stable theories (e.g. algebraically / separably / differentially closed fields, free groups (Sela), planar graphs),
  - *o*-minimal theories (e.g. real closed fields with exponentiation and analytic functions restricted to $[0, 1]$),
  - ordered abelian groups (Gurevich, Schmitt),
  - algebraically closed valued fields, p-adics.

New examples of VC-families

- Examples of NIP theories:
  - stable theories (e.g. algebraically / separably / differentially closed fields, free groups (Sela), planar graphs),
  - *o*-minimal theories (e.g. real closed fields with exponentiation and analytic functions restricted to $[0, 1]$),
  - ordered abelian groups (Gurevich, Schmitt),
  - algebraically closed valued fields, p-adics.
- Non-examples: the theory of the random graph, pseudo-finite fields, ...

Model-theoretic compression schemes

- Given a formula $\phi(x, y)$ and a set of parameters A, a ϕ-type $p(x)$ over A is a maximal consistent collection of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for $a \in A$.

Model-theoretic compression schemes

- Given a formula $\phi(x, y)$ and a set of parameters A, a ϕ-type $p(x)$ over A is a maximal consistent collection of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for $a \in A$.
- A type $p(x) \in S_\phi(A)$ is *definable* if there is some $\psi(y, z) \in L$ and $b \in A^{|b|}$ such that for any $a \in A$, $\phi(x, a) \in p \Leftrightarrow \psi(a, b)$ holds.

Model-theoretic compression schemes

- Given a formula $\phi(x, y)$ and a set of parameters A, a ϕ-type $p(x)$ over A is a maximal consistent collection of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for $a \in A$.
- A type $p(x) \in S_\phi(A)$ is *definable* if there is some $\psi(y, z) \in L$ and $b \in A^{|b|}$ such that for any $a \in A$, $\phi(x, a) \in p \Leftrightarrow \psi(a, b)$ holds.
- We say that ϕ-types are uniformly definable if $\psi(y, z)$ can be chosen independently of A and p.

Model-theoretic compression schemes

- Given a formula $\phi(x, y)$ and a set of parameters A, a ϕ-type $p(x)$ over A is a maximal consistent collection of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for $a \in A$.
- A type $p(x) \in S_\phi(A)$ is *definable* if there is some $\psi(y, z) \in L$ and $b \in A^{|b|}$ such that for any $a \in A$, $\phi(x, a) \in p \Leftrightarrow \psi(a, b)$ holds.
- We say that ϕ-types are uniformly definable if $\psi(y, z)$ can be chosen independently of A and p.
- Definability of types over arbitrary sets is a characteristic property of stable theories, and usually fails in NIP (consider $(\mathbb{Q}, <)$).

Model-theoretic compression schemes

- Given a formula $\phi(x, y)$ and a set of parameters A, a ϕ-type $p(x)$ over A is a maximal consistent collection of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for $a \in A$.
- A type $p(x) \in S_\phi(A)$ is *definable* if there is some $\psi(y, z) \in L$ and $b \in A^{|b|}$ such that for any $a \in A$, $\phi(x, a) \in p \Leftrightarrow \psi(a, b)$ holds.
- We say that ϕ-types are uniformly definable if $\psi(y, z)$ can be chosen independently of A and p.
- Definability of types over arbitrary sets is a characteristic property of stable theories, and usually fails in NIP (consider $(\mathbb{Q}, <)$).
- Laskowski observed that uniform definability of types over *finite* sets implies Warmuth conjecture (and is essentially a model-theoretic version of it).

Model-theoretic compression schemes

Theorem

[Ch., Simon] If T is NIP*, then for any formula $\phi(x, y)$, ϕ-types are uniformly definable over finite sets. This implies that every uniformly definable family of sets in an* NIP *structure admits a compression scheme.*

Model-theoretic compression schemes

Theorem

[Ch., Simon] If T is NIP*, then for any formula $\phi(x, y)$, ϕ-types are uniformly definable over finite sets. This implies that every uniformly definable family of sets in an* NIP *structure admits a compression scheme.*

- Note that we require not only the family $\mathcal{F}$ itself to be of bounded VC-dimension, but also certain families produced from it in a definable way, and that the bound on the size of the compression scheme is not constructive.

Model-theoretic compression schemes

Theorem

[Ch., Simon] If T is NIP, *then for any formula $\phi(x, y)$, ϕ-types are uniformly definable over finite sets. This implies that every uniformly definable family of sets in an* NIP *structure admits a compression scheme.*

- Note that we require not only the family $\mathcal{F}$ itself to be of bounded VC-dimension, but also certain families produced from it in a definable way, and that the bound on the size of the compression scheme is not constructive.
- Main ingredients of the proof:
  - invariant types, indiscernible sequences, honest definitions in NIP (all these tools are quite infinitary),
  - careful use of logical compactness,
  - The (p, q)-theorem.

Transversals and the (p, q)-theorem

Definition

We say that $\mathcal{F}$ satisfies the (p, q)-*property*, where $p \geq q$, if for every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq p$ there is some $\mathcal{F}'' \subseteq \mathcal{F}'$ with $|\mathcal{F}''| \geq q$ such that $\bigcap \{A \in \mathcal{F}''\} \neq \emptyset$.

Transversals and the (p, q)-theorem

Definition

We say that $\mathcal{F}$ satisfies the (p, q)-*property*, where $p \geq q$, if for every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq p$ there is some $\mathcal{F}'' \subseteq \mathcal{F}'$ with $|\mathcal{F}''| \geq q$ such that $\bigcap \{A \in \mathcal{F}''\} \neq \emptyset$.

Fact

Assume that $p \geq q > d$. Then there is an $N = N(p, q)$ such that if $\mathcal{F}$ is a finite family of subsets of X of finite VC-codimension d and satisfies the (p, q)-property, then there are $b_0, \ldots, b_N \in X$ such that for every $A \in \mathcal{F}$, $b_i \in A$ for some $i < N$.

Transversals and the (p, q)-theorem

Definition

We say that $\mathcal{F}$ satisfies the (p, q)-*property*, where $p \geq q$, if for every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq p$ there is some $\mathcal{F}'' \subseteq \mathcal{F}'$ with $|\mathcal{F}''| \geq q$ such that $\bigcap\{A \in \mathcal{F}''\} \neq \emptyset$.

Fact

Assume that $p \geq q > d$. Then there is an $N = N(p, q)$ such that if $\mathcal{F}$ is a finite family of subsets of X of finite VC-codimension d and satisfies the (p, q)-property, then there are $b_0, \ldots, b_N \in X$ such that for every $A \in \mathcal{F}$, $b_i \in A$ for some $i < N$.

- Was proved for families of convex subsets of the Euclidian space by Alon and Kleitman solving a long-standing open problem

Transversals and the (p, q)-theorem

Definition

We say that $\mathcal{F}$ satisfies the (p, q)-*property*, where $p \geq q$, if for every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq p$ there is some $\mathcal{F}'' \subseteq \mathcal{F}'$ with $|\mathcal{F}''| \geq q$ such that $\bigcap \{A \in \mathcal{F}''\} \neq \emptyset$.

Fact

Assume that $p \geq q > d$. Then there is an $N = N(p, q)$ such that if $\mathcal{F}$ is a finite family of subsets of X of finite VC-codimension d and satisfies the (p, q)-property, then there are $b_0, \ldots, b_N \in X$ such that for every $A \in \mathcal{F}$, $b_i \in A$ for some $i < N$.

- Was proved for families of convex subsets of the Euclidian space by Alon and Kleitman solving a long-standing open problem
- Then for families of finite VC- dimension by Matousek (combining ε-nets with the existence of fractional Helly numbers for VC-families)

Transversals and the (p, q)-theorem

Definition

We say that $\mathcal{F}$ satisfies the (p, q)-*property*, where $p \geq q$, if for every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq p$ there is some $\mathcal{F}'' \subseteq \mathcal{F}'$ with $|\mathcal{F}''| \geq q$ such that $\bigcap \{A \in \mathcal{F}''\} \neq \emptyset$.

Fact

Assume that $p \geq q > d$. Then there is an $N = N(p, q)$ such that if $\mathcal{F}$ is a finite family of subsets of X of finite VC-codimension d and satisfies the (p, q)-property, then there are $b_0, \ldots, b_N \in X$ such that for every $A \in \mathcal{F}$, $b_i \in A$ for some $i < N$.

- Was proved for families of convex subsets of the Euclidian space by Alon and Kleitman solving a long-standing open problem
- Then for families of finite VC- dimension by Matousek (combining ε-nets with the existence of fractional Helly numbers for VC-families)
- Closely connected to a finitary version of forking from model theory.

Set theory: counting cuts in linear orders

- There are some questions of descriptive set theory character around VC-dimension and generalizations of PAC learning (Pestov), but I'll concentrate on connections to cardinal arithmetic.

Set theory: counting cuts in linear orders

- There are some questions of descriptive set theory character around VC-dimension and generalizations of PAC learning (Pestov), but I'll concentrate on connections to cardinal arithmetic.
- Let κ be an *infinite* cardinal.

Definition

$\operatorname{ded} \kappa = \sup\{|I|\colon I \text{ is a linear order with a dense subset of size } \leq \kappa\}$.

Set theory: counting cuts in linear orders

- There are some questions of descriptive set theory character around VC-dimension and generalizations of PAC learning (Pestov), but I'll concentrate on connections to cardinal arithmetic.
- Let κ be an *infinite* cardinal.

Definition

$\operatorname{ded}\kappa = \sup\{|I|\colon\ I$ is a linear order with a dense subset of size $\leq \kappa\}$.

- In general the supremum need not be attained.

Equivalent ways to compute $\operatorname{ded} \kappa$

The following cardinals are the same:

1. $\operatorname{ded} \kappa$,

Equivalent ways to compute ded κ

The following cardinals are the same:

1. ded κ,
2. $\sup\{\lambda$: exists a linear order I of size $\leq \kappa$ with λ Dedekind cuts$\}$,

Equivalent ways to compute ded κ

The following cardinals are the same:

1. ded κ,
2. sup$\{\lambda$: exists a linear order I of size $\leq \kappa$ with λ Dedekind cuts$\}$,
3. sup$\{\lambda$: exists a regular μ and a linear order of size $\leq \kappa$ with λ cuts of *cofinality* μ *on both sides*$\}$
   (by a theorem of Kramer, Shelah, Tent and Thomas),

Equivalent ways to compute ded κ

The following cardinals are the same:

1. ded κ,
2. sup$\{\lambda$: exists a linear order I of size $\leq \kappa$ with λ Dedekind cuts$\}$,
3. sup$\{\lambda$: exists a regular μ and a linear order of size $\leq \kappa$ with λ cuts of *cofinality* μ *on both sides*$\}$ (by a theorem of Kramer, Shelah, Tent and Thomas),
4. sup$\{\lambda$: exists a regular μ and a tree T of size $\leq \kappa$ with λ branches of length $\mu\}$.

Some basic properties of ded κ

- $\kappa < \operatorname{ded} \kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)

Some basic properties of $\operatorname{ded} \kappa$

- $\kappa < \operatorname{ded} \kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded} \aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)

Some basic properties of ded κ

- $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
- Assuming GCH, $\operatorname{ded}\kappa = 2^\kappa$ for all κ.

Some basic properties of $\operatorname{ded}\kappa$

- $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
- Assuming GCH, $\operatorname{ded}\kappa = 2^\kappa$ for all κ.
- [Baumgartner] If $2^\kappa = \kappa^{+n}$ (i.e. the nth sucessor of κ) for some $n \in \omega$, then $\operatorname{ded}\kappa = 2^\kappa$.

Some basic properties of ded κ

- $\kappa < \operatorname{ded} \kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded} \aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
- Assuming GCH, $\operatorname{ded} \kappa = 2^\kappa$ for all κ.
- [Baumgartner] If $2^\kappa = \kappa^{+n}$ (i.e. the nth sucessor of κ) for some $n \in \omega$, then $\operatorname{ded} \kappa = 2^\kappa$.
- So is $\operatorname{ded} \kappa$ the same as 2^κ in general?

Some basic properties of ded κ

- $\kappa < \operatorname{ded} \kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded} \aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
- Assuming GCH, $\operatorname{ded} \kappa = 2^\kappa$ for all κ.
- [Baumgartner] If $2^\kappa = \kappa^{+n}$ (i.e. the nth sucessor of κ) for some $n \in \omega$, then $\operatorname{ded} \kappa = 2^\kappa$.
- So is $\operatorname{ded} \kappa$ the same as 2^κ in general?

Fact

[Mitchell] For any κ with $\operatorname{cf} \kappa > \aleph_0$ it is consistent with ZFC that $\operatorname{ded} \kappa < 2^\kappa$.

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).
- We define $f_T(\kappa) = \sup\{|S_T(M)| : M \models T, |M| = \kappa\}$.

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).
- We define $f_T(\kappa) = \sup \{|S_T(M)| : M \models T, |M| = \kappa\}$.

Fact

[Keisler], [Shelah] For any countable T, f_T is one of the following functions: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded} \kappa$, $(\operatorname{ded} \kappa)^{\aleph_0}$, 2^{κ} *(and each of these functions occurs for some T).*

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).
- We define $f_T(\kappa) = \sup\{|S_T(M)| : M \models T, |M| = \kappa\}$.

Fact

[Keisler], [Shelah] For any countable T, f_T is one of the following functions: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded}\kappa$, $(\operatorname{ded}\kappa)^{\aleph_0}$, 2^κ (and each of these functions occurs for some T).

- These functions are distinguished by combinatorial dividing lines, resp. ω-stability, superstability, stability, non-multi-order, NIP.

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).
- We define $f_T(\kappa) = \sup\{|S_T(M)| : M \models T, |M| = \kappa\}$.

Fact

[Keisler], [Shelah] For any countable T, f_T is one of the following functions: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded}\kappa$, $(\operatorname{ded}\kappa)^{\aleph_0}$, 2^{κ} (and each of these functions occurs for some T).

- These functions are distinguished by combinatorial dividing lines, resp. ω-stability, superstability, stability, non-multi-order, NIP.
- In fact, the last dichotomy is an "infinite Shelah-Sauer lemma" (on finite values, number of brunches in a tree is polynomial) $\Rightarrow$ reduction to 1 variable.

Further properties of $\operatorname{ded}\kappa$

- So we have $\kappa < \operatorname{ded}\kappa \leq (\operatorname{ded}\kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded}\kappa = 2^{\kappa}$ under GCH.

Further properties of ded κ

- So we have $\kappa < \operatorname{ded} \kappa \leq (\operatorname{ded} \kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded} \kappa = 2^{\kappa}$ under GCH.
- [Keisler, 1976] Is it consistent that $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$?

Further properties of $\operatorname{ded} \kappa$

- So we have $\kappa < \operatorname{ded} \kappa \leq (\operatorname{ded} \kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded} \kappa = 2^{\kappa}$ under GCH.
- [Keisler, 1976] Is it consistent that $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$?

Theorem

[Ch., Kaplan, Shelah] It is consistent with ZFC *that* $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$ *for some* κ.

Further properties of ded κ

- So we have $\kappa < \operatorname{ded}\kappa \leq (\operatorname{ded}\kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded}\kappa = 2^{\kappa}$ under GCH.
- [Keisler, 1976] Is it consistent that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$?

Theorem

[Ch., Kaplan, Shelah] It is consistent with ZFC *that* $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ *for some* κ.

- Our proof uses Easton forcing and elaborates on Mitchell's argument. We show that e.g. consistently $\operatorname{ded}\aleph_\omega = \aleph_{\omega+\omega}$ and $(\operatorname{ded}\aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$.

Further properties of ded κ

- So we have $\kappa < \operatorname{ded} \kappa \leq (\operatorname{ded} \kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded} \kappa = 2^{\kappa}$ under GCH.
- [Keisler, 1976] Is it consistent that $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$?

Theorem
[Ch., Kaplan, Shelah] It is consistent with ZFC *that* $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$ *for some* κ.

- Our proof uses Easton forcing and elaborates on Mitchell's argument. We show that e.g. consistently $\operatorname{ded} \aleph_\omega = \aleph_{\omega+\omega}$ and $(\operatorname{ded} \aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$.
- **Problem**. Is it consistent that $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0} < 2^{\kappa}$ at the same time for some κ?

Bounding exponent in terms of ded κ

- Recall that by Mitchell consistently ded $\kappa < 2^{\kappa}$. However:

Bounding exponent in terms of ded κ

- Recall that by Mitchell consistently $\operatorname{ded} \kappa < 2^{\kappa}$. However:

Theorem

[Ch., Shelah] $2^{\kappa} \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded} \kappa)))$ *for all infinite* κ.

Bounding exponent in terms of ded κ

- Recall that by Mitchell consistently ded $\kappa < 2^\kappa$. However:

Theorem
[Ch., Shelah] $2^\kappa \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded}\kappa)))$ *for all infinite* κ.

- The proof uses Shelah's PCF theory.

Bounding exponent in terms of ded κ

- Recall that by Mitchell consistently $\operatorname{ded} \kappa < 2^{\kappa}$. However:

Theorem

[Ch., Shelah] $2^{\kappa} \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded} \kappa)))$ *for all infinite* κ.

- The proof uses Shelah's PCF theory.
- **Problem**. What is the minimal number of iterations which works for all models of ZFC (or for some classes of cardinals)? At least 2, and 4 is enough.

Tame topological dynamics

- Stable group theory: genericity, stabilizers, Hrushovski's reconstruction of groups from generic data (e.g. various generalizations of these are used in his results on approximate subgroups).

Tame topological dynamics

- Stable group theory: genericity, stabilizers, Hrushovski's reconstruction of groups from generic data (e.g. various generalizations of these are used in his results on approximate subgroups).
- Groups definable in *o*-minimal structures: real Lie groups, Pillay's conjecture, etc.

Tame topological dynamics

- Stable group theory: genericity, stabilizers, Hrushovski's reconstruction of groups from generic data (e.g. various generalizations of these are used in his results on approximate subgroups).
- Groups definable in *o*-minimal structures: real Lie groups, Pillay's conjecture, etc.
- Common generalization: study of NIP groups, leads to considering questions of "definable" topological dynamics.

Tame topological dynamics

- Stable group theory: genericity, stabilizers, Hrushovski's reconstruction of groups from generic data (e.g. various generalizations of these are used in his results on approximate subgroups).
- Groups definable in *o*-minimal structures: real Lie groups, Pillay's conjecture, etc.
- Common generalization: study of NIP groups, leads to considering questions of "definable" topological dynamics.
- Parallel program: actions of automorphism groups of ω-categorical theories (recent connections to stability by Ben Yaacov, Tsankov, Ibarlucia) - some things are very similar, but we concentrate on the definable case for now.

Definable actions

- Let $M \models T$ and G is an M-definable group (e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$ etc).

Definable actions

- Let $M \models T$ and G is an M-definable group (e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$ etc).
- G acts by homeomorphisms on $S_G(M)$, its space of types - this is a universal flow with respect to "definable" actions, we try to understand this system: minimal flows, generics, measures, etc.

Definable actions

- Let $M \models T$ and G is an M-definable group (e.g. $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$ etc).
- G acts by homeomorphisms on $S_G(M)$, its space of types - this is a universal flow with respect to “definable” actions, we try to understand this system: minimal flows, generics, measures, etc.

Definition

An action of a definable group G on a compact space X is called *definable* if:

- G acts by homeomorphisms,

Definable actions

- Let $M \models T$ and G is an M-definable group (e.g. $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$ etc).
- G acts by homeomorphisms on $S_G(M)$, its space of types - this is a universal flow with respect to "definable" actions, we try to understand this system: minimal flows, generics, measures, etc.

Definition

An action of a definable group G on a compact space X is called *definable* if:

- G acts by homeomorphisms,
- for each $x \in X$, the map $f_x : G \to X$ taking x to gx is definable (a function f from a definable set $Y \subseteq M$ to X is definable if for any closed disjoint $C_1, C_2 \subseteq X$ there is an M-definable $D \subseteq Y$ such that $f^{-1}(C_1) \subseteq D$ and $D \cap f^{-1}(C_2) = \emptyset$).

Definably amenable groups

- Let $\mathfrak{M}_G(M)$ denote the totally disconnected compact space of probability measures on $S_G(M)$ (we view it as a closed subset of $[0,1]^{L(M)}$ with the product topology, coincides with the weak*-topology).

Definably amenable groups

- Let $\mathfrak{M}_G(M)$ denote the totally disconnected compact space of probability measures on $S_G(M)$ (we view it as a closed subset of $[0,1]^{L(M)}$ with the product topology, coincides with the weak*-topology).
- Now $(G, S_G(M))$ is a universal ambit for the definable actions of G, and G is definably (extremely) amenable iff every definable action admits a G-invariant measure (a G-fixed point).

Definably amenable groups

- Let $\mathfrak{M}_G(M)$ denote the totally disconnected compact space of probability measures on $S_G(M)$ (we view it as a closed subset of $[0,1]^{L(M)}$ with the product topology, coincides with the weak*-topology).
- Now $(G, S_G(M))$ is a universal ambit for the definable actions of G, and G is definably (extremely) amenable iff every definable action admits a G-invariant measure (a G-fixed point).
- Equivalently, G is *definably amenable* if there is a global (left) G-invariant finitely additive measure on the boolean algebra of definable subsets of G (can be extended from clopens in $S_G(M)$ to Borel sets by regularity).

Definably amenable groups

Example

The following groups are definably amenable:

- Any definable group which is amenable as a discrete group (e.g. solvable groups),

Definably amenable groups

Example

The following groups are definably amenable:

- Any definable group which is amenable as a discrete group (e.g. solvable groups),
- Any definably compact group in an *o*-minimal theory (e.g. $SO_3(\mathbb{R})$ *is* definably amenable, despite Banach-Tarski).

Definably amenable groups

Example

The following groups are definably amenable:

- Any definable group which is amenable as a discrete group (e.g. solvable groups),
- Any definably compact group in an *o*-minimal theory (e.g. $SO_3(\mathbb{R})$ *is* definably amenable, despite Banach-Tarski).
- Any stable group. In particular the free group F_2 is known by the work of Sela to be stable as a pure group, and hence is definably amenable.

Definably amenable groups

Example

The following groups are definably amenable:

- Any definable group which is amenable as a discrete group (e.g. solvable groups),
- Any definably compact group in an *o*-minimal theory (e.g. $SO_3(\mathbb{R})$ *is* definably amenable, despite Banach-Tarski).
- Any stable group. In particular the free group F_2 is known by the work of Sela to be stable as a pure group, and hence is definably amenable.
- Any pseudo-finite group.

Definably amenable groups

Example

The following groups are definably amenable:

- Any definable group which is amenable as a discrete group (e.g. solvable groups),
- Any definably compact group in an *o*-minimal theory (e.g. $SO_3(\mathbb{R})$ *is* definably amenable, despite Banach-Tarski).
- Any stable group. In particular the free group F_2 is known by the work of Sela to be stable as a pure group, and hence is definably amenable.
- Any pseudo-finite group.

- If K is an algebraically closed valued field or a real closed field and $n > 1$, then $SL(n, K)$ is not definably amenable.

Connected components

- In an algebraic group over ACF, one can consider a connected component of 1 with repsect to the Zariski topology. In RCF, consider infinitesimals.

Definition

Let A be a small subset of $\mathbb{M}$ (a monster model for T). We define:

- $G^0_A = \bigcap \{H \leq G : H$ is A-definable, of finite index$\}$.
- $G^{00}_A = \bigcap \{H \leq G : H$ is type-definable over A, of bounded index$\}$.

Connected components

- In an algebraic group over ACF, one can consider a connected component of 1 with repsect to the Zariski topology. In RCF, consider infinitesimals.

Definition

Let A be a small subset of $\mathbb{M}$ (a monster model for T). We define:

- $G^0_A = \bigcap \{H \leq G : H$ is A-definable, of finite index$\}$.
- $G^{00}_A =$
  $\bigcap \{H \leq G : H$ is type-definable over A, of bounded index$\}$.

- In general depend on A and can get smaller as A grows.

Connected components

- In an algebraic group over ACF, one can consider a connected component of 1 with repsect to the Zariski topology. In RCF, consider infinitesimals.

Definition

Let A be a small subset of $\mathbb{M}$ (a monster model for T). We define:

- $G_A^0 = \bigcap \{H \leq G : H \text{ is } A\text{-definable, of finite index}\}$.
- $G_A^{00} =$
  $\bigcap \{H \leq G : H \text{ is type-definable over } A, \text{ of bounded index}\}$.

- In general depend on A and can get smaller as A grows.

Fact

Let T be NIP. *Then for every small set A we have:*

- *[Baldwin-Saxl]* $G_\emptyset^0 = G_A^0$,

Connected components

- In an algebraic group over ACF, one can consider a connected component of 1 with repsect to the Zariski topology. In RCF, consider infinitesimals.

Definition

Let A be a small subset of $\mathbb{M}$ (a monster model for T). We define:

- $G_A^0 = \bigcap \{H \leq G : H$ is A-definable, of finite index$\}$.
- $G_A^{00} =$ $\bigcap \{H \leq G : H$ is type-definable over A, of bounded index$\}$.

- In general depend on A and can get smaller as A grows.

Fact

Let T be NIP. *Then for every small set A we have:*

- *[Baldwin-Saxl]* $G_\emptyset^0 = G_A^0$,
- *[Shelah]* $G_\emptyset^{00} = G_A^{00}$,

Connected components

- In an algebraic group over ACF, one can consider a connected component of 1 with repsect to the Zariski topology. In RCF, consider infinitesimals.

Definition

Let A be a small subset of $\mathbb{M}$ (a monster model for T). We define:

- $G_A^0 = \bigcap \{H \leq G : H$ is A-definable, of finite index$\}$.
- $G_A^{00} =$ $\bigcap \{H \leq G : H$ is type-definable over A, of bounded index$\}$.

- In general depend on A and can get smaller as A grows.

Fact

Let T be NIP*. Then for every small set A we have:*

- *[Baldwin-Saxl]* $G_\emptyset^0 = G_A^0$,
- *[Shelah]* $G_\emptyset^{00} = G_A^{00}$,
- *Both are normal* $\mathrm{Aut}(\mathbb{M})$*-invariant subgroups of G of bounded index.*

The logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map.
- We endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.

The logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map.
- We endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.
- With this topology, G/G^{00} is a compact topological group.

The logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map.
- We endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.
- With this topology, G/G^{00} is a compact topological group.
- If $G^0 = G^{00}$ (e.g. G is a stable group), then G/G^{00} is a profinite group: it is the inverse image of the groups G/H, where H ranges over all definable subgroups of finite index. E.g. If $G = (Z, +)$, then $G^{00} = G^0$ is the set of elements divisible by all n. The quotient G/G^{00} is isomorphic as a topological group to $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$.

The logic topology on G/G^{00}

- Let $\pi : G \to G/G^{00}$ be the quotient map.
- We endow G/G^{00} with the *logic topology*: a set $S \subseteq G/G^{00}$ is closed iff $\pi^{-1}(S)$ is type-definable over some (any) small model M.
- With this topology, G/G^{00} is a compact topological group.
- If $G^0 = G^{00}$ (e.g. G is a stable group), then G/G^{00} is a profinite group: it is the inverse image of the groups G/H, where H ranges over all definable subgroups of finite index. E.g. If $G = (Z, +)$, then $G^{00} = G^0$ is the set of elements divisible by all n. The quotient G/G^{00} is isomorphic as a topological group to $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$.
- If $G = \mathrm{SO}(2, \mathcal{R})$ is the circle group defined in a real closed field $\mathcal{R}$, then G^{00} is the set of infinitesimal elements of G and G/G^{00} is canonically isomorphic to the standard circle group $SO(2, \mathbb{R})$. Note also that $G^0 = G$, so $\neq G^{00}$.

Some results for definably amenable NIP groups (joint work with Pierre Simon)

- Ergodic measures are liftings of the Haar measure on G/G^{00} via certain invariant types.

Some results for definably amenable NIP groups (joint work with Pierre Simon)

- Ergodic measures are liftings of the Haar measure on G/G^{00} via certain invariant types.
- There is a coherent theory of genericity extending the stable case.

Some results for definably amenable NIP groups (joint work with Pierre Simon)

- Ergodic measures are liftings of the Haar measure on G/G^{00} via certain invariant types.
- There is a coherent theory of genericity extending the stable case.
- Proofs use VC theory along with forking calculus in NIP theories.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.
- Also $(G, E(X))$ is a flow as well, G acts on $E(X)$ by $\pi_g \circ f$.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.
- Also $(G, E(X))$ is a flow as well, G acts on $E(X)$ by $\pi_g \circ f$.
- The minimal closed left ideals in $(E(X), \cdot)$ coincide with the minimal subflows of $(G, E(X))$ (nonempty closed subset I of S such that $a \cdot I \subseteq I$ for all $a \in E(X)$).

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.
- Also $(G, E(X))$ is a flow as well, G acts on $E(X)$ by $\pi_g \circ f$.
- The minimal closed left ideals in $(E(X), \cdot)$ coincide with the minimal subflows of $(G, E(X))$ (nonempty closed subset I of S such that $a \cdot I \subseteq I$ for all $a \in E(X)$).
- For any closed left ideal I, there is an idempotent $u \in I$.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.
- Also $(G, E(X))$ is a flow as well, G acts on $E(X)$ by $\pi_g \circ f$.
- The minimal closed left ideals in $(E(X), \cdot)$ coincide with the minimal subflows of $(G, E(X))$ (nonempty closed subset I of S such that $a \cdot I \subseteq I$ for all $a \in E(X)$).
- For any closed left ideal I, there is an idempotent $u \in I$.
- If I is minimal and $u \in I$ idempotent, then $u \cdot I$ is a group.

Ellis group

- Let (G, X) be a dynamical system, and for $g \in G$ let $\pi_g : X \to X$ be the corresponding homeomorphism.
- Let $E(X)$ be the closure of $\{\pi_g(x) : g \in G\}$ in the compact space X^X.
- Then $(E(X), \cdot)$, where $\cdot$ is composition, is a semigroup (called the Ellis enveloping semigroup of (G, X)).
- Note: $E(X)$ is a compact Hausdorff topological space such that $\cdot$ is continuous in the first coordinate, namely for each $b \in E(X)$ the map taking x to $x \cdot b$ is continuous.
- Also $(G, E(X))$ is a flow as well, G acts on $E(X)$ by $\pi_g \circ f$.
- The minimal closed left ideals in $(E(X), \cdot)$ coincide with the minimal subflows of $(G, E(X))$ (nonempty closed subset I of S such that $a \cdot I \subseteq I$ for all $a \in E(X)$).
- For any closed left ideal I, there is an idempotent $u \in I$.
- If I is minimal and $u \in I$ idempotent, then $u \cdot I$ is a group.
- Moreover, as u, I vary, these groups are isomorphic.

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.
- There is a natural surjective group homomorphism $\pi : u \cdot I \to G/G^{00}$. Newelski conjectured that in NIP, it is an isomorphism. But $SL(2, \mathbb{R})$ is a counterexample.

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.
- There is a natural surjective group homomorphism $\pi : u \cdot I \to G/G^{00}$. Newelski conjectured that in NIP, it is an isomorphism. But $SL(2, \mathbb{R})$ is a counterexample.
- **Corrected Ellis group conjecture** [Pillay]. Suppose G is a definably amenable NIP group. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot I$ is an isomorphism, for some/any minimal subflow I of $S_G(M_0)$ and idempotent $u \in I$ (i.e. π is injective).

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.
- There is a natural surjective group homomorphism $\pi : u \cdot I \to G/G^{00}$. Newelski conjectured that in NIP, it is an isomorphism. But $SL(2, \mathbb{R})$ is a counterexample.
- **Corrected Ellis group conjecture** [Pillay]. Suppose G is a definably amenable NIP group. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot I$ is an isomorphism, for some/any minimal subflow I of $S_G(M_0)$ and idempotent $u \in I$ (i.e. π is injective).
- Some partial results (including the *o*-minimal case) in a joint work [Ch., Pillay, Simon].

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.
- There is a natural surjective group homomorphism $\pi : u \cdot I \to G/G^{00}$. Newelski conjectured that in NIP, it is an isomorphism. But $SL(2, \mathbb{R})$ is a counterexample.
- **Corrected Ellis group conjecture** [Pillay]. Suppose G is a definably amenable NIP group. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot I$ is an isomorphism, for some/any minimal subflow I of $S_G(M_0)$ and idempotent $u \in I$ (i.e. π is injective).
- Some partial results (including the o-minimal case) in a joint work [Ch., Pillay, Simon].

Theorem

[Ch., Simon] The Ellis group conjecture is true.

Ellis group conjecture

- Applying this construction to our definable group G acting on the space of its types, $(G, S_G(M))$, we obtain some Ellis group $u \cdot I$.
- There is a natural surjective group homomorphism $\pi : u \cdot I \to G/G^{00}$. Newelski conjectured that in NIP, it is an isomorphism. But $SL(2, \mathbb{R})$ is a counterexample.
- **Corrected Ellis group conjecture** [Pillay]. Suppose G is a definably amenable NIP group. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot I$ is an isomorphism, for some/any minimal subflow I of $S_G(M_0)$ and idempotent $u \in I$ (i.e. π is injective).
- Some partial results (including the *o*-minimal case) in a joint work [Ch., Pillay, Simon].

Theorem

[Ch., Simon] The Ellis group conjecture is true.

- We can recover G/G^{00} abstractly from the action and the Ellis group does not depend on the model of T.

Ellis group conjecture

- Main ingredients of the proof:
  - fine analysis of Borel definability of invariant types in NIP theories,
  - generic compact domination for the Baire ideal (a more general version of the unique ergodicity for *tame* minimal systems of Glasner, in the definable category).