

MATH 115A (CHERNIKOV), SPRING 2016
PROBLEM SET 6
DUE FRIDAY, MAY 13

Problem 1. Do Exercise 1, Section 2.4. Justify each answer.

Problem 2.

- (1) Which of the following pairs of vector spaces are isomorphic? Justify your answers.
 - (a) F^3 and $P_3(F)$.
 - (b) F^4 and $P_3(F)$.
 - (c) $M_{2 \times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$.
 - (d) $V = \{A \in M_{2 \times 2}(\mathbb{R}) : \text{tr}(A) = 0\}$ and \mathbb{R}^4 .
- (2) Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$. Construct an isomorphism from V to F^3 .

Problem 3. Do Exercise 1, Section 2.5. Justify each answer.

Problem 4. For each of the following pairs of ordered bases β and β' for V , find the change of coordinates matrix that changes β' -coordinates into β -coordinates.

- (1) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$, in $V = \mathbb{R}^2$.
- (2) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$, in $V = \mathbb{R}^2$.
- (3) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$, in $V = \mathbb{R}^2$.
- (4) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$, in $V = P_2(\mathbb{R})$.

Problem 5. Given two matrices $A, B \in M_{n \times n}(F)$, we say that B is *similar* to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Similarity is an equivalence relation.

Recall that the trace of a matrix $A \in M_{n \times n}(F)$ is the sum of its diagonal entries, that is $\text{tr}(A) = A_{1,1} + A_{2,2} + \dots + A_{n,n}$. Prove the following statements:

- (1) For any $A, B \in M_{n \times n}(F)$, $\text{tr}(AB) = \text{tr}(BA)$.
- (2) If A and B are similar, then $\text{tr}(A) = \text{tr}(B)$.
- (3) Let V be a vector space with $\dim(V) = n$, and let β, β' be two ordered bases for V , and let $T \in \mathcal{L}(V)$ be arbitrary. Then $\text{tr}([T]_{\beta}) = \text{tr}([T]_{\beta'})$.

Problem 6. Prove the following generalization of Theorem 2.23.

Let $T : V \rightarrow W$ be a linear transformation, $\dim(V), \dim(W) < \infty$. Let β and β' be ordered bases for V , and let γ and γ' be ordered bases for W . Then

$$[T]_{\beta'}^{\gamma'} = P^{-1} [T]_{\beta}^{\gamma} Q,$$

where Q is the matrix that changes β' coordinates into β -coordinates, and P is the matrix that changes γ' -coordinates into γ -coordinates.

Problem 7. Compute the determinants of the following matrices (and provide the details of your calculations).

$$(1) \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix},$$

$$(2) \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix},$$

$$(3) \begin{pmatrix} 14 & 80 & -14 & -76 & -4 \\ 0 & 2 & 1 & 3 & 0 \\ 1 & 0 & -2 & 2 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 \end{pmatrix}.$$

Problem 8. Prove that $\det(cA) = c^n \det(A)$ for any $A \in M_{n \times n}(F)$ and $c \in F$.