Properties of lin. transformations (contd.)

Example.

1) Let $T: F^n \to F^{n-1}$ be defined by $T((a_1,\dots,a_n)) = (a_1,\dots,a_{n-1})$. — so T "forgets" the n-th component.
Then T is linear, $N(T) = \{(\underbrace{0,\dots,0}, a_n) : a_n \in F\}$ and $R(T) = F^{n-1}$.
And $\dim(F^n) = n$, $\dim(N(T)) = 1^{n-1}$ and $\dim(R(T)) = \dim(F^{n-1}) = n-1$.
2) Let $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ be the differentiation transformation, that is $T(p(x)) = p'(x)$ for any polynomial $p(x)$.
Then $T(p(x)) = 0 \iff p'(x) = 0 \iff p(x)$ constant. So $N(T) = \{$ constant polynomials in $P(\mathbb{R})\}$.
Recall that $\{1, x, x^2, \dots, x^{n-1}\}$ is a basis for $P_{n-1}(\mathbb{R})$. Since $1 = T(x)$, $x = \frac{1}{2}T(x^2), \dots, x^{n-1} = \frac{1}{n}T(x^n)$, it follows that $V = \text{Span}(\{T(x), \dots, T(x^n)\}) = R(T)$.
Thus $\dim(P_n(\mathbb{R})) = n+1$, $\dim(R(T)) = n$ and $\dim(N(T)) = 1$.

Definition. Let $T: V \to W$ be a lin. transf.
T is **injective** if $T(v) = T(u)$ implies $v = u$, for all $u, v \in V$.
T is **surjective** if for every $w \in W$ there is some $v \in V$ such that $T(v) = w$.
T is **bijective** if it is both injective and surjective.

Theorem 2.4 Let $T: V \to W$ be linear. Then T is injective if and only if $N(T) = \{0\}$.

Proof.
"$\Rightarrow$" Suppose T is injective, and let $x \in N(T)$.
Then $T(x) = 0 = T(0) \Rightarrow x = 0$. Hence $N(T) = \{0\}$.
"$\Leftarrow$". Assume $N(T) = \{0\}$ and suppose $T(x) = T(y)$.
Then $0 = T(x) - T(y) = T(x-y)$, as T is lin.
So $x - y \in N(T) = \{0\}$. Hence $x - y = 0$, or $x = y$.

Theorem 2.5. Let $T: V \to W$ be lin, and $\dim(V) = \dim(W) < \infty$. Then the following are equivalent:
a) T is injective.
b) T is surjective.
c) T is bijective.
d) $\dim(R(T)) = \dim(V)$.

Proof.
By the dimension theorem, $\dim(N(T)) + \dim(R(T)) = \dim(V)$.
We have:
T is injective $\overset{\text{(Theorem 2.4)}}{\iff}$ $N(T) = \{0\} \iff \dim(N(T)) = 0 \iff \dim(R(T)) = \dim(V) \iff$
$\iff \dim(R(T)) = \dim(W) \underset{\text{(Thm 1.11)}}{\iff} R(T) = W \iff T$ is surjective.

Example.
1) Define $T: F^2 \to F^2$ by $T(a_1, a_2) = (a_1 + a_2, a_1)$.
Then $N(T) = \{0\}$, so T is injective. By Theorem 2.5, T is also surjective.
2) Define $T: P_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ by $T(a_0 + a_1x + \dots + a_nx^n) = (a_0, a_1, \dots, a_n)$.
Then T is linear and injective, hence T is bijective (as $\dim(P_n(\mathbb{R})) = \dim(\mathbb{R}^{n+1})$!).

Next we show that every lin. transf. is completely determined by its action on a basis!

Theorem 2.6.
Let V, W be v.s. over a field F, and let $\{v_1, \dots, v_n\}$ be a basis for V.
For any $w_1, \dots, w_n \in W$ there exists **exactly one** lin. transformation $T: V \to W$ s.t. $T(v_i) = w_i$ for $i = 1, \dots, n$.

Proof.

Let $x \in V$. Then $x = \sum_{i=1}^{n} a_i v_i$ for some unique scalars $a_1, \dots, a_n \in F$. (because $\{v_1, \dots, v_n\}$ is a basis!)
We define a map $T: V \to W$ by

$$T(x) = \sum_{i=1}^{n} a_i w_i.$$

a) T is linear.
Suppose $u, v \in V$ and $d \in F$. We can write
$u = \sum_{i=1}^{n} b_i v_i$ and $v = \sum_{i=1}^{n} c_i v_i$ for some scalars $b_1, \dots, b_n, c_1, \dots, c_n \in F$.
Then

$$du + v = \sum_{i=1}^{n} (d b_i + c_i) v_i.$$

So

$$T(du+v) = \sum_{i=1}^{n} (d b_i + c_i) w_i = d \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i = d\, T(u) + T(v).$$

b) $T(v_i) = w_i$ for $i = 1, \dots, n$ — clear from the definition of T.

c) T is unique.
Suppose that $U: V \to W$ is linear, and that it also satisfies $U(v_i) = w_i$ for $i = 1, \dots, n$.
Then, for $x \in V$ with $x = \sum_{i=1}^{n} a_i v_i$ we have (as U is linear):

$$U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = T(x).$$

Hence $U = T$.

Corollary. Let V, W be v.s.; V has a finite basis $\{v_1, \dots, v_n\}$.
If $U, T: V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, \dots, n$ then $U = T$.

Example. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the lin. transformation defined by

$$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$$

Suppose that $U: \mathbb{R}^2 \to \mathbb{R}^2$ is any lin. transf.
If we know that $U(1,2) = (3,3)$ and $U(1,1) = (1,3)$, then $U = T$.
This follows from the corollary, because $\{(1,2),(1,1)\}$ is a basis for $\mathbb{R}^2$.

The matrix representation of a lin. transformation.

Definition Let V be a fin. dim. v.s. An ordered basis for V is a basis for V endowed with a specific order.

Example.
- In F^3, $\beta = \{e_1, e_2, e_3\}$ is an ordered basis. (recall $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$).
  Also, $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis.
  So β and γ is the same set, but $\beta \neq \gamma$ as ordered bases. The choice of the order matters!
- For the v.s. F^n, we call $\{e_1, e_2, \dots, e_n\}$ the standard ordered basis for F^n.
- Similarly, for the v.s. $P_n(F)$ we call $\{1, x, \dots, x^n\}$ the standard ordered basis for $P_n(F)$.

Definition. Let $\beta = \{u_1, \dots, u_n\}$ be an ordered basis for a fin. dim. v.s. V.
For $x \in V$, let $a_1, \dots, a_n \in F$ be the unique scalars such that (by Theorem 1.8.)

$$x = \sum_{i=1}^{n} a_i u_i.$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \qquad (\text{so } [x]_\beta \text{ is a vector in } F^n).$$

so each vector can be described by its coordinates with respect to a fixed basis.

- Notice that $[u_i]_\beta = e_i$.
- The correspondence $x \to [x]_\beta$ is a lin. transformation from V to F^n. (Exercise).

Example. Let $V = P_2(\mathbb{R})$, and let $\beta = \{1, x, x^2\}$ be the standard ordered basis for V.
Consider $f(x) = 4 + 6x - 7x^2 \in V$, then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Definition. Suppose V, W are fin. dim. v.s., with ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$, respectively. Let $T: V \to W$ be linear.
Then for each j, $1 \le j \le n$, there exist unique scalars $a_{ij} \in F$, $1 \le i \le m$ such that
$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ for $1 \le j \le n$.

We call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ, and write $A = [T]_\beta^\gamma$.
If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

Notice. • the j^{th} column of A is $[T(v_j)]_\gamma$.
• If $U: V \to W$ is a lin. transf. s.t. $[U]_\beta^\gamma = [T]_\beta^\gamma$ then $U = T$ (by the corollary to Theorem 2.6)
• So $[T]_\beta^\gamma$ gives an explicit way to describe T which is very useful in computations.

Example. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the lin. transf. defined by
$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$.
Let $\beta = \{e_1, e_2\}$, $\gamma = \{e_1, e_2, e_3\}$ – the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Now:
$T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3$
$T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$.
Hence
$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}.$$
But if we take $\gamma' = \{e_3, e_2, e_1\}$, then $[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$.

Definition Let $T, U: V \to W$ be functions, where V, W are v.s. over F, and let $a \in F$. We define:
$T + U: V \to W$ by $(T+U)(x) = T(x) + U(x)$ for all $x \in V$.
$aT: V \to W$ by $(aT)(x) = aT(x)$ for all $x \in V$.
So $T+U$ and aT are both functions from V to W.

These operations preserve linearity.

Theorem 2.7 Let V, W be v.s. over F, let $T, U: V \to W$ be linear.
a) For all $a \in F$, $aT + U$ is linear.
b) With this operations of addition and scalar multiplication, the set of all linear transformations from V to W is a v.s. over F.

Proof.
a) Let $x, y \in V$ and $c \in F$. Then
$\underset{\text{function}}{(aT+U)}\,\underset{\text{vector in } V}{(cx+y)} = (aT)(cx+y) + U(cx+y) = a(T(cx+y)) + (cU(x) + U(y)) = a(cT(x) + T(y)) + cU(x) + U(y) =$
$= acT(x) + cU(x) + aT(y) + U(y) = c(aT+U)(x) + (aT+U)(y)$.
Hence the map $aT + U$ is linear.
b) Note that the zero transformation T_0 (recall $T_0(x) = 0$ for all $x \in V$) plays the role of the zero vector, and its easy to verify that all of the axioms (VS1)-(VS8) of a vector space are satisfied.

Definition For V, W v.s. over F, we denote $\mathcal{L}(V, W) = \{T : T \text{ is a lin. transf. from } V \text{ to } W\}$ – a v.s. over F.
In case $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, W)$.