Eigenvalues and eigenvectors.

Definition. 1) A lin. operator T on a fin. dim. v.s. V is called *diagonalizable* if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. That is,

$$[T]_\beta = \begin{pmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{nn} \end{pmatrix} \quad \text{for some } A_{11}, \dots, A_{nn} \in F.$$

2) A matrix $A \in M_{n\times n}(F)$ is *diagonalizable* if A is *similar* to a diagonal matrix.
Recall: two matrices $A, B \in M_{n\times n}(F)$ are *similar* if there is an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^{-1}AQ$.

Theorem. Let $T: V \to V$ be a lin. operator, $\dim(V) < \infty$ and β, γ ordered bases for V.
Then $\det([T]_\beta) = \det([T]_\gamma)$.

Proof.
There exists an invertible matrix Q s.t. $[T]_\gamma = Q^{-1}[T]_\beta Q$ (namely, the change of coordinates matrix $[I_V]_\gamma^\beta$ converting γ-coordinates to β-coordinates).
Then, using the basic properties of det, we have:
$\det([T]_\gamma) = \det(Q^{-1}[T]_\beta Q) = \det(Q^{-1})\cdot\det([T]_\beta)\cdot\det(Q) = (\det Q)^{-1}\cdot\det Q\cdot\det([T]_\beta) =$
$= \det([T]_\beta)$.

Definition. For a lin. operator T, we define its *determinant*, *$\det T$*, as follows:
choose *any* ordered basis β for V and take $\det T = \det([T]_\beta)$.
(by the previous theorem, the choice of β doesn't matter).

Proposition
a) T is bijective $\iff \det T \neq 0$.
b) T is bijective $\implies \det(T^{-1}) = (\det T)^{-1}$.
c) If $U: V \to V$ is another lin. operator on V, then $\det(TU) = \det T \cdot \det U$.

Proof
Exercise, follows from the analogous properties of the matrix determinant.

Theorem. Let $T: V \to V$ be a lin. operator, $\dim(V) < \infty$, β an ordered basis for V. Then:
T is diagonalizable $\iff [T]_\beta$ is a diagonalizable matrix.

Proof.
Let $\beta = \{v_1, \dots, v_n\}$.
$\Rightarrow$) Assume that T is diagonalizable. This means that there is an ordered basis γ for V such that $D = [T]_\gamma$ is a diagonal matrix. Let $[I_V]_\gamma^\beta$ be the change of coordinates matrix.
Then $[T]_\gamma = Q^{-1}[T]_\beta Q$, so $[T]_\gamma$ and $[T]_\beta$ are similar, so $[T]_\beta$ is diagonalizable.
$\Leftarrow$ Exercise.

Corollary. $A \in M_{n\times n}(F)$ is diagonalizable $\iff L_A$ is diagonalizable.

Problem. When is A/T diagonalizable?

Theorem. T is diagonalizable $\iff$ there is an ordered basis $\beta = \{v_1, \dots, v_n\}$ for V and scalars $\lambda_1, \dots, \lambda_n \in F$ such that
$T(v_j) = \lambda_j v_j$ for $1 \le j \le n$.

Proof.
If $D = [T]_\beta$ is a diagonal matrix, then for each vector $v_j \in \beta$ we have
$T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j$, where $\lambda_j = D_{jj}$.
Conversely, if β is an ord. basis for V s.t. $T(v_j) = \lambda_j v_j$, then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

This argument motivates the following definition.

Definition.

1) A non-zero vector $v \in V$ is an **eigenvector** of T if $T(v) = \lambda v$ for some $\lambda \in F$. We call λ the **eigenvalue** of T corresponding to the eigenvector v.
2) Let $A \in M_{n\times n}(F)$. A non-zero $v \in F^n$ is an **eigenvector** of A if $Av = \lambda v$ for some $\lambda \in F$. And λ is the **eigenvalue** of A corresponding to the eigenvector v.
3) The elements in a basis β as in the last theorem are eigenvectors, and the λ_i's are the respective eigenvalues.

Theorem 5.2. A scalar $\lambda \in F$ is an eigenvalue of $T \iff \det(T - \lambda I_V) = 0$

Proof. We have

$\lambda \in F$ is an eigenvalue of $T \iff T(v) = \lambda v$ for some $v \neq 0$ in $V \iff \overbrace{(T - \lambda I_V)}^{\text{lin. operator on } V}(v) = 0$ for some $v \neq 0$ in $V \iff N(T - \lambda I_V) \neq \{0\} \overset{\text{Thm 2.4, 2.5}}{\iff} T - \lambda I_V$ is not bijective $\underset{\text{properties of det.}}{\iff} \det(T - \lambda I_V) = 0$.

Corollary. Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of $A \iff \det(A - \lambda I_n) = 0$.

Example. Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

Then $\det(A - \lambda I_2) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = (\lambda - 3)(\lambda + 1)$.

Hence by the corollary, the eigenvalues of A are the solutions to $(\lambda-3)(\lambda+1) = 0$ — which are $3, -1$.

Definition 1) The polynomial $f(t) = \det(A - tI_n)$ in the variable t is called the **characteristic polynomial** of A.

2) Given a lin. operator $T: V \to V$, $\dim(V) < \infty$, and β an ordered basis for V, we define the **characteristic polynomial** of T to be the char. polynomial of $A = [T]_\beta$:

$$f(t) = \det(A - tI).$$

Note. Similar matrices have the same char. polynomial, so is well defined.

Properties of char. polynomial.

Let $A \in M_{n\times n}(F)$ be given, and let $f(t)$ be its char. polynomial.

1) $f(t)$ is a polynomial of **degree n** with leading coefficient $(-1)^n$:
$f(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \ldots + c_0$ for some $c_0, \ldots, c_{n-1} \in F$.
2) A scalar $\lambda \in F$ is an eigenvalue of $A \iff f(\lambda) = 0$.
3) A has at most n distinct eigenvalues (as $f(t)$ has at most n roots).
4) If $\lambda \in F$ is an eigenvalue of A, then a vector $x \in F^n$ is an eigenvector of A corresp. to $\lambda \iff x \neq 0$ and $x \in N(L_A - \lambda I_{F^n})$.

Example. Let's consider $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ again, and let's find it

1) The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.
2) Let $B_1 = A - \lambda_1 I_2 = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$.

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ is an eigenvector of A corresponding to λ_1 — by (4) above.

$\iff x \neq 0$ and $x \in N(L_{B_1}) \iff x \neq 0$ and $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\iff$

$\Leftrightarrow \begin{pmatrix} -2x_1+x_2 \\ 4x_1-2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The set of all solutions to this system of lin. equations is
$\left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\}$.

Hence $x \in \mathbb{R}^2$ is an eigenvector corresp. to $\lambda_1 = 3$ $\Leftrightarrow$ $x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for some $t \neq 0$.

3) Let $B_2 = A - \lambda_1 I_2 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$. Hence:
$x \in \mathbb{R}^2$ is an e.vec. of A corresp. to λ_2 $\Leftrightarrow$ $x \neq 0$ and $x \in N(L_{B_2})$ $\Leftrightarrow$ $B_2 \cdot x = 0$ $\Leftrightarrow$ $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
$\Leftrightarrow \begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$.

Hence $N(L_{B_2}) = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\}$. Thus x is an e.vec. corresp. to $\lambda_2 = -1$ $\Leftrightarrow$ $x = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for some $t \neq 0$.

Notice that $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ consisting of e.vectors of A. Thus L_A, and hence A, is diagonalizable.

Determining eigenvectors and eigenvalues of a lin. operator

Let V be a v.s., $\dim(V) = n$. Let β be an ordered basis for V.
Let $T \in \mathcal{L}(V)$ be a lin. operator on V.
Summarizing the results of the previous section, we describe how to determine the e.val's and the e.vec's of T.

1) Determine the matrix representation $[T]_\beta$ of T.
2) Determine the e.val's of T.
 $\lambda \in F$ is an e.val of T $\Leftrightarrow$ λ is a root of the char. polynomial of T.
 That is, we need to find the solutions $x \in F$ of $\det([T]_\beta - x I_n) = 0$.
 There are at most n distinct solutions $\lambda_1, \dots, \lambda_n$.
3) Now for each e.val. λ of T, we can determine the corresponding e.vec's. We have:
 $T(v) = \lambda v \Leftrightarrow (T - \lambda I_V)(v) = 0 \Leftrightarrow [T - \lambda I_V]_\beta [v]_\beta = 0$.
 Therefore, eigenvectors corresponding to λ are the solutions of this system of linear equations. (more precisely, solving this system we find the β-coordinates $[v]_\beta$, which then determines v).

Theorem 5.5

Let $T \in \mathcal{L}(V)$ be a lin. operator on V, and let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \dots, v_k$ are e.vect's of T s.t. v_i corresponds to λ_i, then the set
$\{v_1, \dots, v_k\}$
is lin. indep.

Proof.
By induction on k.
$k=1$. As $v_1 \neq 0$, $\{v_1\}$ is lin. indep.
Induction step, $k-1 \Rightarrow k$.
Suppose we know the theorem for $k-1$ distinct eigenvalues, and lets prove it for k.
Suppose $a_1, \dots, a_k \in F$ are such that
$a_1 v_1 + \dots + a_k v_k = 0$.
Applying the linear transf. $T - \lambda_k I_V$ to both sides and using linearity, we get:
$(T - \lambda_k I_V) = T(0) - \lambda_k I_V(0) = 0 - 0 = 0$.
as $T(v_i) = \lambda_i v_i$
$(T - \lambda_k I_V)(a_1 v_1 + \dots + a_k v_k) = (a_1 T(v_1) + \dots + a_k T(v_k)) - \lambda_k(a_1 v_1 + \dots + a_k v_k) = a_1\lambda_1 v_1 + \dots + a_k \lambda_k v_k -$
$- \lambda_k a_1 v_1 + \dots + \lambda_k a_k v_k = a_1(\lambda_1 - \lambda_k)v_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$.
By Induction Hypothesis, $\{v_1, \dots, v_{k-1}\}$ are lin. indep., so
$a_1(\lambda_1 - \lambda_k) = \dots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$.

Since $\lambda_1, \dots, \lambda_k$ are distinct by assumption, $\lambda_i - \lambda_k \neq 0$ for $i = 1, \dots, k-1$.
Thus $a_1 = \dots = a_{k-1} = 0$.
Hence $a_k v_k = 0$. But as $v_k \neq 0$ (as an eigenvector), we get $a_k = 0$.

Corollary Let $T \in \mathcal{L}(V)$ and $\dim(V) = n$.
If T has n distinct e.val's, then T is diagonalizable.
Proof.
Let $\lambda_1, \dots, \lambda_n$ be n distinct e.val's of T. For each i, let v_i be an eigenvector corresp. to λ_i. By the theorem, $\{v_1, \dots, v_n\}$ is lin. indep. Since $\dim(V) = n$, this set is a basis for V.
Thus V has a basis consisting of eigenvectors for T, so T is diagonalizable.

Ex The converse of Thm 5.5 is false.
For example, the identity operator I_V has only one eigenvalue, namely $\lambda = 1$. However it is diagonalizable!

Def A polynomial $f(t) \in P(F)$ splits over F if there are scalars $c, a_1, \dots, a_n \in F$ (not necessarily distinct) such that
$f(t) = c(t-a_1)(t-a_2)\cdot \ldots \cdot (t-a_n)$.

Ex 1) $t^2 - 1 \in P_2(\mathbb{R})$ splits over $\mathbb{R}$, namely $t^2 - 1 = (t-1)(t+1)$.
2) $t^2 + 1 \in P_2(\mathbb{R})$ doesn't split over $\mathbb{R}$.
However, viewed as a polynomial in $P_2(\mathbb{C})$, it splits over $\mathbb{C}$: $t^2 + 1 = (t+i)(t-i)$.

Thm 5.6 The char. polynomial of any diagonalizable lin. operator splits.
Proof.
Let $n = \dim(V)$, $T \in \mathcal{L}(V)$ be diagonalizable, then there is an ordered basis β for V s.t. $[T]_\beta = D$, where D is of the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Let $f(t)$ be the char. polynomial of T. Then

$$f(t) = \det(D - tI) = \begin{pmatrix} \lambda_1 - t & & 0 \\ & \lambda_2 - t & \\ 0 & & \ddots \;\; \lambda_n - t \end{pmatrix} = (\lambda_1 - t) \cdot \ldots \cdot (\lambda_n - t) = (-1)^n (t - \lambda_1) \cdot \ldots \cdot (t - \lambda_n).$$

Def Let λ be an e.val. of a lin. operator or matrix with char. polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$. (That is, $f(t)$ can be written as $f(t) = (t-\lambda)^k g(t)$ for some polynomial $g(t)$).

Ex Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$, then $f(t) = -(t-3)^2(t-4)$. Hence $\lambda = 3$ is an e.val. of A with mult. 2, and $\lambda = 4$ is an e.val. of A with mult. 1.

Def Let $T \in \mathcal{L}(V)$, λ an eigenvalue of T. We define E_λ, the eigenspace of T corresp. to λ, as

$E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - \lambda I_V)$. (and similary for a matrix).
Note that this is a subspace of V, consisting of 0 and the e.vects of T corresp to λ.

Thm 5.7. Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$, λ an e.val. of T with multiplicity m.
Then $1 \le \dim(E_\lambda) \le m$.

Proof.
Choose an ordered basis $\{v_1, \dots, v_p\}$ for E_λ.
By the replacement thm, can extend it to an ordered basis $\beta = \{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}$ for V.
Let $A = [T]_\beta$.
Notice that v_i, $i = 1, \dots, p$, is an eigenvector of T corresp to λ. Hence

$$A = \left(\begin{array}{c|c} \lambda I_p & B \\ \hline 0 & C \end{array}\right).$$

Then

$$f(t) = \det(A - tI_n) = \det \left(\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - tI_{n-p} \end{array}\right) \overset{\text{exercise!}}{=} \det((\lambda - t) I_p) \det(C - tI_{n-p}) =$$
$$= (\lambda - t)^p g(t),$$

where $g(t)$ is a polynomial.
Thus $(\lambda - t)^p$ is a factor of $f(t)$, hence the mult. of λ is at least p. But $\dim(E_\lambda) = p$, so $\dim(E_\lambda) \le m$.

Lemma Let $T \in \mathcal{L}(V)$, $\lambda_1, \dots, \lambda_k$ distinct e.vals of T.
Let $v_i \in E_{\lambda_i}$ for each $i = 1, \dots, k$.
If $v_1 + \dots + v_k = 0$ then $v_i = 0$ for all i.

Pf Suppose otherwise, say we have $v_i \ne 0$ for $1 \le i \le m$, and $v_i = 0$ for $i > m$, for some $1 \le m \le k$.
Then for each $i \le m$, v_i is an e.vect of T corresp. to λ_i. (as $v_i \in E_{\lambda_i} \setminus \{0\}$),
and $v_1 + \dots + v_m = 0$.
But this contradicts Thm 5.5 as $v_1, \dots, v_m$ must be lin. indep. Therefor $v_i = 0$ for all $i = 1, \dots, k$.

Thm 5.8 Let $T \in \mathcal{L}(V)$, let $\lambda_1, \dots, \lambda_k$ be distinct e.vals of T.
For each $i = 1, \dots, k$, let S_i be a finite lin. indep. subset of E_{λ_i}.
Then $S = S_1 \cup \dots \cup S_k$ is also a lin. indep. subset of V.

Proof.
Suppose that $S_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ for each $i = 1, \dots, k$.
Then $S = \{v_{i,j} : 1 \le j \le n_i, 1 \le i \le k\}$.
Let $\{a_{i,j}\}$ be any scalars in F s.t.

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0.$$

For each i, let $w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}$. Then: $w_i \in \overset{\text{subspace!}}{E_{\lambda_i}}$, and $w_1 + \dots + w_k = 0$.
By the lemma, $w_i = 0$ for all $i = 1, \dots, k$.
But as each S_i is indep., it follow that $a_{ij} = 0$ for all j.
Hence S is lin. indep.