MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 1

DUE THURSDAY, APRIL 13

Problem 1. Let V denote the set of all pairs of real numbers, that is $V = \{(a, b) : a, b \in \mathbb{R}\}$. For all (a_1, a_2) and (b_1, b_2) elements of V and $c \in \mathbb{R}$, we define:

(1) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ (the usual operation of addition),
(2) $c\,(a_1, a_2) = (ca_1, a_2)$.

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Problem 2. Prove that the following statements are true in any vector space V over a field F.

(1) $0x = 0$ for each $x \in V$.
(2) $(-a)\,x = -\,(ax) = a\,(-x)$ for each $a \in F$ and each $x \in V$.
(3) $a0 = 0$ for each $a \in F$ (where $0 \in V$ is the zero-vector).

(Say explicitly which of the axioms (VS1)–(VS8) you are using on each step of the proof).

Problem 3. Recall that $\mathbb{R}^2$ is the vector space with addition and scalar multiplication given by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $a\,(x_1, x_2) = (ax_1, ax_2)$.

(1) Give an example of a subset of $\mathbb{R}^2$ which is closed under addition, but not under scalar multiplication.
(that is, a set $S \subseteq \mathbb{R}^2$ such that for any two vectors from S their sum is also in S, but there is some $a \in \mathbb{R}$ and $(x, y) \in S$ such that $a\,(x, y)$ is not in S).
(2) Give an example of a subset of $\mathbb{R}^2$ which is closed under scalar multiplication, but is not closed under addition.
(that is, a set $S \subseteq \mathbb{R}^2$ such that for any $a \in \mathbb{R}$ and any $(x, y) \in S$, the vector $a\,(x, y)$ is also in S, but there are some $(x_1, y_1), (x_2, y_2) \in S$ such that their sum is not in S).

Problem 4. Determine whether the following sets are subspaces of $\mathbb{R}^3$. Justify your answer (if it is a subspace, prove it; if not, explain which condition fails).

(1) $W_1 = \big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\big\}$,
(2) $W_2 = \big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 5a_3\big\}$,
(3) $W_3 = \big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 a_2 a_3 = 0\big\}$,
(4) $W_4 = \big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 + 3a_3 = 0\big\}$.

Problem 5. Let S be a non-empty set and F a field, and let $\mathcal{F}(S, F)$ be the vector space of all functions from S to F. Prove that for any element $s_0 \in S$ the set $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ is a subspace of $\mathcal{F}(S, F)$.

Problem 6. We denote by $M_{m\times n}(F)$ the set of all $m\times n$ matrices with entries from a field F. So every element $A\in M_{m\times n}$ is of the form

$$A=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix},$$

with $A_{ij}\in F$ for all $1\le i\le m, 1\le j\le n$. The entries A_{ij} with $i=j$ are called the *diagonal entries* of the matrix. $M_{m\times n}$ is a vector space over F with the following operations matrix addition and scalar multiplication: for $A,B\in M_{m\times n}(F)$ and $c\in F$, we define the matrices $A+B$ and cA by taking $(A+B)_{ij}=A_{ij}+B_{ij}$ and $(cA)_{ij}=cA_{ij}$.

(1) Show that $M_{m\times n}(\mathbb{R})$ satisfies (VS3), (VS7) and (VS8).
(2) Let W_1 be the set of all diagonal matrices in $M_{n\times n}(\mathbb{R})$ (a matrix $A=(A_{ij}:1\le i,j\le n)$ is called *diagonal* if all its entries outside of the diagonal are zero, that is $A_{ij}=0$ whenever $i\ne j$). Show that W_1 is a subspace of $M_{n\times n}(\mathbb{R})$.
(3) Let W_2 be the set of all matrices in $M_{m\times n}(\mathbb{R})$ with non-negative entries (that is, $A_{ij}\ge 0$ for all $1\le i\le m, 1\le j\le n$). Show that W_2 is not a subspace of $M_{m\times n}(\mathbb{R})$.

Problem 7. Prove that a subset W of a vector space V is a subspace of V if and only if $W\ne\emptyset$, and, whenever $a\in F$ and $x,y\in W$, then $ax\in W$ and $x+y\in W$. (Hint: use Theorem 1.3)

Problem 8.

(1) Let V be the vector space $\mathbb{R}^2$. Give an example of two subspaces W_1 and W_2 of V such that their union $W_1\cup W_2$ is not a subspace of V.
(2) Let now V be an arbitrary vector space, and let W_1 and W_2 be subspaces of V. Show that $W_1\cup W_2$ is a subspace of V if and only if $W_1\subseteq W_2$ or $W_2\subseteq W_1$.