MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 5

DUE THURSDAY, MAY 11

Problem 1. Do Exercise 1, Section 2.2. Justify each answer.

Problem 2. Do Exercise 1, Section 2.3. Justify each answer.

Problem 3. Let V, W, Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(1) Prove that if UT is injective, then T is injective. Must U also be injective? Justify your answer.
(2) Prove that if UT is surjective, then U is surjective. Must T also be surjective? Justify your answer.
(3) Prove that if U and T are bijective, then UT is also bijective.

Problem 4. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be linear transformations defined by

$$T(f(x)) = f'(x)\, g(x) + 2f(x)$$

and

$$U\left(a + bx + cx^2\right) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(1) Compute $[U]_\beta^\gamma$, $[T]_\beta$. Compute $[UT]_\beta^\gamma$ in two ways: directly and using Theorem 2.11.
(2) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (1) and Theorem 2.14 to verify your result.

Problem 5. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
(2) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.
(3) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.
(4) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$.
(5) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)\, x^2$.
(6) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c + d \end{pmatrix}$.

Problem 6. Let $A, B \in M_{n\times n}(F)$ be given. Show:

(1) If A and B are invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
(2) If A is invertible, then A^t is invertible, and $(A^t)^{-1} = \left(A^{-1}\right)^t$.
(3) If $AB = I_n$, then A and B are invertible, and $A = B^{-1}$, $B = A^{-1}$.
(4) If $A^2 = 0$, then A is not invertible.

Problem 7. Let V, W be finite dimensional vector spaces, and let T be an isomorphism. Let V_0 be a subspace of V. Show that $T(V_0)$ is a subspace of W, and that $\dim(V_0) = \dim(T(V_0))$.

Problem 8. Let $T : V \to W$ be a linear transformation, $\dim(V) = n$, $\dim(W) = m$. Let β and γ be ordered bases for V and W, respectively.

Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$. (Hint: use the previous problem.)