MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 7

DUE THURSDAY, MAY 25

Problem 1. Do Exercise 1, Section 4.4. Justify each answer.

Problem 2. Do Exercise 1, Section 5.1. Justify each answer.

Problem 3. Let V be an n-dimensional vector space, and suppose that $T \in \mathcal{L}(V)$ is invertible. Determine the characteristic polynomial of T^{-1} in terms of the characteristic polynomial of T.

Problem 4. Let V be a finite dimensional vector space. Prove that a linear operator $T \in \mathcal{L}(V)$ is invertible if and only if 0 is not an eigenvalue of T.

Problem 5. For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

(1) $V = \mathbb{R}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$.

(2) $V = P_1(\mathbb{R})$, $T(a + bx) = (6a - 6b) + (12a - 11b)\,x$, and $\beta = \{3 + 4x, 2 + 3x\}$.

(3) $V = M_{2\times 2}(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$ and

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

Problem 6. Determine the eigenvalues and eigenvectors for each of the following matrices in $M_{3\times 3}(\mathbb{R})$.

(1) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$.

(2) $B = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$.

Problem 7. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(1) $V = \mathbb{R}^2$ and $T(a, b) = (-2a + 3b, -10a + 9b)$.

(2) $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)\,x + f(3)$.

(3) $V = M_{2\times 2}(\mathbb{R})$ and $T(A) = A^t + 2\,\mathrm{tr}(A) \cdot I_2$.

Problem 8. Suppose that $T \in \mathcal{L}(V)$ is such that every non-zero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.