MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 8

DUE THURSDAY, JUNE 1

Problem 1. Do Exercise 1, Section 5.2, parts (a) – (g). Justify each answer.

Problem 2. For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, determine if A is diagonalizable. If A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(1) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(2) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$,

(3) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$,

(4) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$.

Problem 3. For each of the following linear operators T on a vector space V, determine if T is diagonalizable. If T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(1) $V = \mathbb{R}^3$ and T is defined by $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$.

(2) $V = P_2(\mathbb{R})$ and T is defined by $T\left(ax^2 + bx + c\right) = cx^2 + bx + a$.

(3) $V = P_3(\mathbb{R})$ and T is defined by $T(f(x)) = f'(x) + f''(x)$ (where $f'(x)$ and $f''(x)$ are the 1st and the 2nd derivatives of $f(x)$, respectively).

(4) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$.

Problem 4. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find A^{1000}.

(Hint: reduce the problem to raising a diagonal matrix to the 1000th power).

Problem 5. Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Problem 6. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Problem 7. Let T be an invertible linear operator on a vector space V.

(1) Prove that a scalar $\lambda \in F$ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(2) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(3) Prove that if T is diagonalizable, then T^{-1} is also diagonalizable.

Problem 8. Let $A \in M_{n\times n}(F)$.

(1) Prove that A and A^t have the same characteristic polynomial
(2) It follows from (1) that A and A^t share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.
Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.
(3) Prove that if A is diagonalizable, then A^t is also diagonalizable.