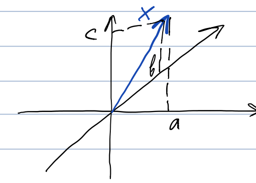
Example

Consider $\mathbb{R}^3$ with the standar inner product.
Then for $x=(a,b,c)\in\mathbb{R}^3$, the length of x is given by
$\sqrt{a^2+b^2+c^2} = \sqrt{\langle x,x\rangle}$.



By imitating what happens in $\mathbb{R}^3$, we can define length in an arbitrary inner product space.

Def Let V be an inner product space.
For any $x\in V$, we define the norm or length of x by $\|x\| = \sqrt{\langle x,x\rangle}$.

Example

Let $V=F^n$. If $x=(a_1,\dots,a_n)$, then
$$\|x\| = \|(a_1,\dots,a_n)\| = \left(\sum_{i=1}^n |a_i|^2\right)^{\frac{1}{2}}$$
is the Euclidean definition of length.

Many properties of the Euclidean length in $\mathbb{R}^3$ hold in general.

Thm 6-2

Let V be an inner product space over F. Then for all $x,y\in V$ and $c\in F$ we have:

a) $\|cx\| = |c|\cdot\|x\|$.

b) $\|x\|=0 \iff x=0$. (and $\|x\|\geq 0$ for any x).

c) $|\langle x,y\rangle| \leq \|x\|\cdot\|y\|$. (Cauchy – Schwarz Inequality)

d) $\|x+y\| \leq \|x\|+\|y\|$ (Triangle Inequality)

Proof.

c) If $y=0$, then $\langle x,y\rangle=0$ and $\|y\|=0$, so the result holds.
Assume now $y\neq 0$.
For any $c\in F$ we have
$$0\leq \|x-cy\|^2 = \langle x-cy, x-cy\rangle = \langle x, x-cy\rangle - c\langle y, x-cy\rangle = \langle x,x\rangle - \bar{c}\langle x,y\rangle - c\langle y,x\rangle + c\bar{c}\langle y,y\rangle.$$
In particular, if we take $c = \dfrac{\langle x,y\rangle}{\underbrace{\langle y,y\rangle}_{\neq 0 \text{ as } y\neq 0}}$, we have

We are using that
$a\cdot\bar{a} = |a|^2$ and $a+\bar{a} = 2\,\mathrm{Re}(a)$
for any complex number a, and that
$\langle x,y\rangle = \overline{\langle y,x\rangle}$

$$0\leq \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle} = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$, and (c) follows.

d) We have
$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle = \|x\|^2 + 2\,\overbrace{\mathrm{Re}\,\langle x,y\rangle}^{\text{the real part of the complex number } \langle x,y\rangle} + \|y\|^2 \leq$$
$$\leq \|x\|^2 + 2\,|\langle x,y\rangle| + \|y\|^2 \underset{\text{by (c)}}{\leq} \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2.$$

Orthogonality

- As you may recall from earlier courses, for $\mathbb{R}^2$ and $\mathbb{R}^3$ there is another formula expressing the dot product of two vectors x and y:
  $\langle x,y\rangle = \|x\|\cdot\|y\|\cdot\cos\theta$
  where θ $(0\leq\theta\leq\pi)$ is the angle between x and y.
  Notice also that non-zero vectors are perpendicular if and only if $\cos\theta = 0$, that is if and only if $\langle x,y\rangle = 0$.
- We generalize this to define perpendicularity in arbitrary inner product spaces.

Def. Let V be an inner product space.
1) Vectors x and y in V are orthogonal (perpendicular) if $\langle x, y\rangle = 0$.
2) A subset S of V is orthogonal if any two distinct vectors in S are orthogonal.
3) A vector x in V is a unit vector if $\|x\| = 1$.
4) A subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.

Remark (Normalization)
1) $S = \{v_1, v_2, \dots\}$ is orthonormal $\Leftrightarrow$ $\begin{cases} \langle v_i, v_j\rangle = 0 & \text{for all } i \neq j \\ \langle v_i, v_i\rangle = 1 & \text{for all } i. \end{cases}$

2) If $S = \{v_1, v_2, \dots\}$ is orthogonal, and $a_i \in F$ are any non-zero scalars, then the set $\{a_1v_1, a_2v_2, \dots\}$ is also orthogonal (as $0 = \langle a_iv_i, a_jv_j\rangle = a_i\bar{a}_j\langle v_i, v_j\rangle \Leftrightarrow \langle v_i, v_j\rangle = 0$).

3) If x is any non-zero vector in V, then $y := \left(\frac{1}{\|x\|}\right)x$ is a unit vector. We say that y is obtained from x by normalizing.

4) In view of (2) and (3), we can obtain an orthonormal set from an orthogonal set by normalizing every vector in it.

Ex. In $\mathbb{R}^3$, $\{(1,1,0), (1,-1,1), (-1,1,2)\}$ is an orthogonal set of non-zero vectors, but it is not orthonormal.
Normalizing each of the vectors, we obtain an orthonormal set
$\left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\right\}$.

Orthonormal bases and Gram – Schmidt orthogonalization

Def Let V be an inner product space.
A subset S of V is an orthonormal basis for V if S is an ordered basis for V and S is orthonormal.

- Just as bases are the building blocks of vector spaces, orthonormal bases are the building blocks of inner product spaces.

Ex The standard ordered basis for F^n is an orthonormal basis for the inner product space F^n (with the standard inner product).

Ex The set $\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$ is an orthonormal basis for $\mathbb{R}^2$.

Importance of orthonormal sets and bases is illustrated by the following theorem and its corollaries

Thm 6.3 Let V be an inner prod. space and $S = \{v_1, \dots, v_k\}$ an orthogonal subset of V consisting of non-zero vectors.
If $y \in \mathrm{Span}(S)$, then $$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2} v_i.$$

Proof. Write $y = \sum_{i=1}^{k} a_iv_i$, where $a_1, \dots, a_k \in F$. Then, for $1 \le j \le k$, we have
$$\langle y, v_j\rangle = \left\langle \sum_{i=1}^{k} a_iv_i, v_j \right\rangle = \sum_{i=1}^{k} a_i \underbrace{\langle v_i, v_j\rangle}_{\substack{=0 \text{ for all } i\neq j \text{ by} \\ \text{orthogonality}}} = a_j\langle v_j, v_j\rangle = a_j\|v_j\|^2.$$

So $a_j = \dfrac{\langle y, v_j\rangle}{\|v_j\|^2}$.

Corollary 1 If, in addition to the hypotheses of Thm. 6.3, S is orthonormal and $y \in \mathrm{Span}(S)$, then
$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i .$$

Corollary 2. Let V be an inner product space, and let S be an orthogonal subset of V consisting of non-zero vectors. Then S is lin. indep.

Proof.
Suppose that $v_1, \dots, v_k \in S$ and $\sum_{i=1}^{k} a_i v_i = 0$. As in the proof of Thm 6.3 with $y=0$, we have $a_j = \frac{\langle 0, v_j \rangle}{\|v_j\|^2} = 0$ for all j. So S is lin. indep.

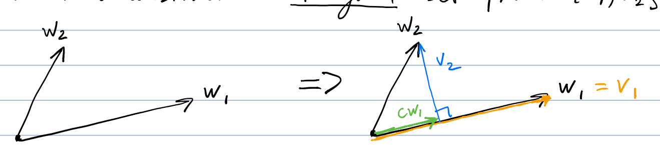
Ex By Corollary 2, the orthonormal set $\beta = \{ \underset{v_1}{\frac{1}{\sqrt{2}}(1,1,0)}, \underset{v_2}{\frac{1}{\sqrt{3}}(1,-1,1)}, \underset{v_3}{\frac{1}{\sqrt{6}}(-1,1,2)} \}$ from a prev. example is an orthonormal basis for $\mathbb{R}^3$.
Let $x = (2,1,3)$. Using Corollary 1, it is easy to calculate the coordinates of x relatively to β:

$a_1 = \langle x, v_1 \rangle = 2 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} + 3 \cdot 0 = \frac{3}{\sqrt{2}}$, $\quad a_2 = 2 \cdot \frac{1}{\sqrt{3}} - 1 \cdot \frac{1}{\sqrt{3}} + 3 \cdot \frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$,

$a_3 = 2 \cdot \left(-\frac{1}{\sqrt{6}}\right) + 1 \cdot \frac{1}{\sqrt{6}} + 3 \cdot \frac{2}{\sqrt{6}} = \frac{5}{\sqrt{6}}$. $\quad$ Hence $x = \frac{3}{\sqrt{2}} v_1 + \frac{4}{\sqrt{3}} v_2 + \frac{5}{\sqrt{6}} v_3$.

- So it is useful to have an orthonormal basis.
  But we still need to show that it always exists! (in a fin. dim. inner product space)

Ex Let's consider a simple case first.
- Suppose $\{w_1, w_2\}$ is a lin. indep subset of an inner product space (and hence a basis for $W = \mathrm{Span}(\{w_1, w_2\})$).
- We want to construct an orthogonal set from $\{w_1, w_2\}$ that spans the same subspace W.



The picture above suggests that the set $\{v_1, v_2\}$ with $v_1 = w_1$, $v_2 = w_2 - cw_1$ has this property if c is chosen so that v_2 is orthogonal to w_1.
To find c, we need to solve the equation
$0 = \langle v_2, w_1 \rangle = \langle w_2 - cw_1, w_1 \rangle = \langle w_2, w_1 \rangle - c \langle w_1, w_1 \rangle$.
So $c = \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2}$, and $v_2 = w_2 - \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2} w_1$

The next theorem show that this process can be extended to any finite lin. indep. subset.

Thm 6.4 Let V be an inner prod. space and $S = \{w_1, \dots, w_n\}$ a lin. indep subset of V.
Define $S' = \{v_1, \dots, v_n\}$, where $v_1 = w_1$ and
$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad \text{for } 2 \le k \le n. \qquad (1)$$
Then S' is an orthogonal set of non-zero vectors such that $\mathrm{Span}(S') = \mathrm{Span}(S)$.

Proof. By induction on n, the number of vectors in S.
For $k = 1, 2, \dots, n$, let $S_k = \{w_1, \dots, w_k\}$.
If $n=1$, then the theorem is proved by taking $S_1' = S_1$, i.e. $v_1 = w_1 \neq 0$.
For $n>1$. Assume that the set
$S_{k-1}' = \{v_1, \dots, v_{k-1}\}$ with the desired properties has been constructed by the repeated use of (1).

We show that $S'_k = \{v_1, \dots, v_{k-1}, v_k\}$ also has the desired properties, where v_k is obtained from S'_{k-1} by (1).

If $v_k = 0$, then (1) implies that $w_k \in \mathrm{Span}(S'_{k-1}) \overset{\text{inductive hyp.}}{=} \mathrm{Span}(S_{k-1})$, which contradicts the assumption that S_k is lin. indep. Hence $v_k \neq 0$.

For $1 \le i \le k-1$ it follows from (1) that

$$\langle v_k, v_i \rangle = \langle w_k, v_i \rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} \underbrace{\langle v_j, v_i \rangle}_{\substack{=0 \text{ for } i \neq j \text{ by ind. hyp. on} \\ \text{orthogonality of } S'_{k-1}.}} = \langle w_k, v_i \rangle - \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} \|v_i\|^2 = 0$$

Hence S'_k is an orthogonal set of non-zero vectors.

By (1), $\mathrm{Span}(S'_k) \subseteq \mathrm{Span}(S_k)$. By Cor. 2 to Thm 6.3, S'_k is lin. indep.

So $\dim(\mathrm{Span}(S'_k)) = \dim(\mathrm{Span}(S_k)) = k$. Therefore $\mathrm{Span}(S'_k) = \mathrm{Span}(S_k)$.

The construction of $\{v_1, \dots, v_n\}$ by the use of Thm 6.4 is called the Gram-Schmidt process.

Ex Let $V = \mathbb{R}^4$, with the standard inner prod., $w_1 = (1,0,1,0)$, $w_2 = (1,1,1,1)$, $w_3 = (0,1,2,1)$. Then $\{w_1, w_2, w_3\}$ is lin. indep.

We use the G-S process to compute the orthogonal vectors v_1, v_2, v_3

Take $v_1 = w_1 = (1,0,1,0)$. Then

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = (1,1,1,1) - \frac{(1\cdot 1 + 0\cdot 1 + 1\cdot 1 + 0\cdot 1)}{\left(\sqrt{1^2+0^2+1^2+0^2}\right)^2}(1,0,1,0) = (0,1,0,1).$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 = (0,1,2,1) - \frac{2}{2}(1,0,1,0) - \frac{2}{2}(0,1,0,1) = (-1,0,1,0).$$

Now we normalize them to obtain the orthonormal basis $\{u_1, u_2, u_3\}$ where

$$u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}}(0,1,0,1)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}(-1,0,1,0).$$

Thm 6.5 Let V be a non-zero inner prod. space, $\dim(V) < \infty$.

Then V has an orthonormal basis β.

Furthermore, if $\beta = \{v_1, \dots, v_n\}$ and $x \in V$, then $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$.

Proof.

Let β_0 be an ordered basis for V.

Applying Thm 6.4, we obtain an orthogonal set β' of non-zero vectors with $\mathrm{Span}(\beta_0) = \mathrm{Span}(\beta') = V$.

Normalizing each vector in β', we obtain an orthonormal set β with $\mathrm{Span}(\beta) = \mathrm{Span}(\beta') = V$.

By Corollary 2 (to Thm 6.3), β is lin. indep — hence an orthonormal basis for V. The rest follows by Corollary 1.

Cor Let V be an inner prod. space with an orthonormal basis $\beta = \{v_1, \dots, v_n\}$.

Let T be a lin. operator on V, and let $A = [T]_\beta$.

Then for any i, j, $A_{ij} = \langle T(v_j), v_i \rangle$.

Orthogonal complement

Def Let $S \subseteq V$ be non-empty, V – inner prod. space.

Let $S^\perp$ ("S perp") be the set of all vectors in V that are orthogonal to every vector in S. That is,

$S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. – the orthogonal complement of S.

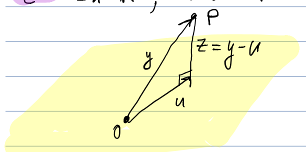
Note: $S^\perp$ is a subspace of V for any $S \subseteq V$.

Ex

1) $\{0\}^\perp = V$ and $V^\perp = \{0\}$ for any inner prod. space

2) If $V = \mathbb{R}^3$ and $S = \{e_3\}$, then $S^\perp$ equals the xy-plane.

Ex In $\mathbb{R}^3$, consider a point P and a plane W. How to find the distance from P to W?



By the picture, can be restated as:
Determine the vector $u \in W$ that is "closest" to y, the distance given by $\|y-u\|$. Notice: $z = y-u$ is orthogonal to every vector in W, so $z \in W^\perp$.

We can find u as follows.

Thm 6.6 Let W be a subspace of an inner prod. space V, $\dim(W) < \infty$. Let $y \in V$. Then there exists unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{v_1, \dots, v_k\}$ is an orthonormal basis for W, then $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$.

Proof. Let $\{v_1, \dots, v_k\}$ and u be as above. (see HW).
Let $z = y - u$. Then $u \in W$ and $y = u + z$.
To show that $z \in W^\perp$, it suffices to show that z is orthogonal to each v_j.
For any j we have:

$$\langle z, v_j \rangle = \left\langle \left(y - \sum_{i=1}^{k} \langle y, v_i \rangle \right), v_j \right\rangle = \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \langle v_i, v_j \rangle =$$

$$= \langle y, v_j \rangle - \langle y, v_j \rangle = 0.$$

To show uniqueness of u and z, suppose that $y = u + z = u' + z'$, where $u' \in W$, $z' \in W^\perp$.
Then $u - u' = z' - z \in W \cap W^\perp = \{0\}$. Therefore, $u = u'$ and $z = z'$.

Corollary In the notation of Thm 6.6, the vector u is the unique vector in W that is "closest" to y. That is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and the equality holds if and only if $x = u$.

Proof
See Textbook, p. 350.

This vector u in the corollary is called the orthogonal projection of y on W.

Thm 6.7 Let $S = \{v_1, \dots, v_k\} \subseteq V$ be orthonormal, $\dim(V) = n$. Then

a) S can be extended to an orthonormal basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V.

b) If $W = \mathrm{Span}(S)$, then $S_1 = \{v_{k+1}, \dots, v_n\}$ is an orthonormal basis for $W^\perp$.

c) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^\perp)$.

Proof

a) By Cor. 2 to the replacement thm, S can be extended to an ordered basis $S' = \{v_1, \dots, v_k, w_{k+1}, \dots, w_n\}$ for V. Now apply the Gram-Schmidt process to S'.
The first k vectors resulting from this process are the vectors in S, and this new set spans V.
Normalize the last $n-k$ vectors.

b) Because S_1 is a subset of a basis, it is lin. indep.
Since S_1 is clearly a subset of $W^\perp$, we need only show that $\mathrm{Span}(S_1) = W^\perp$.
For any $x \in V$,
$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$.

If $x \in W^\perp$, then $\langle x, v_i \rangle = 0$ for $1 \le i \le k$. Therefore,

$$x = \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \in \mathrm{Span}(S_1).$$

c) Let W be a subspace of V. It is a fin. dim. inner prod. space because V is, so has an orthonorm. basis $\{v_1, \dots, v_k\}$. By (a) and (b),

$$\dim(V) = n = k + (n-k) = \dim(W) + \dim(W^\perp).$$

Ex.

Let $W = \mathrm{Span}(\{e_1, e_2\})$ in F^3. Then $x = (a,b,c) \in W^\perp \iff 0 = \langle x, e_1 \rangle = a$ and $0 = \langle x, e_2 \rangle = b$. So $x = (0,0,c)$, and $W^\perp = \mathrm{Span}(\{e_3\})$.