The matrix representation of a lin. transformation.

Definition Let V be a fin. dim. v.s. An ordered basis for V is a basis for V endowed with a specific order.

Example. • In F^3, $\beta = \{e_1, e_2, e_3\}$ is an ordered basis. (recall $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$).
Also, $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis.
So β and γ is the same set, but $\beta \neq \gamma$ as ordered bases. The choice of the order matters!

• For the v.s. F^n, we call $\{e_1, e_2, \dots, e_n\}$ the standard ordered basis for F^n.

• Similarly, for the v.s. $P_n(F)$ we call $\{1, x, \dots, x^n\}$ the standard ordered basis for $P_n(F)$.

Definition. Let $\beta = \{u_1, \dots, u_n\}$ be an ordered basis for a fin. dim. v.s. V.
For $x \in V$, let $a_1, \dots, a_n \in F$ be the unique scalars such that (by Theorem 1.8.)

$$x = \sum_{i=1}^{n} a_i u_i .$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \qquad (\text{so } [x]_\beta \text{ is a vector in } F^n).$$

so each vector can be described by its coordinates with respect to a fixed basis.

• Notice that $[u_i]_\beta = e_i$.

• The correspondence $x \to [x]_\beta$ is a lin. transformation from V to F^n. (Exercise).

Example. Let $V = P_2(\mathbb{R})$, and let $\beta = \{1, x, x^2\}$ be the standard ordered basis for V.
Consider $f(x) = 4 + 6x - 7x^2 \in V$, then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Definition. Suppose V, W are fin. dim. v.s., with ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$, respectively.
Let $T : V \to W$ be linear.
Then for each j, $1 \le j \le n$, there exist unique scalars $a_{ij} \in F$, $i \le i \le m$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for} \quad 1 \le j \le n.$$

We call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ, and write $A = [T]_\beta^\gamma$.
If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

Notice. • the j^{th} column of A is $[T(v_j)]_\gamma$.

• If $U : V \to W$ is a lin. transf. s.t. $[U]_\beta^\gamma = [T]_\beta^\gamma$ then $U = T$ (by the corollary to Theorem 2.6)

• So $[T]_\beta^\gamma$ gives an explicit way to describe T which is very useful in computations.

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the lin. transf. defined by

$$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2).$$

Let $\beta = \{e_1, e_2\}$, $\gamma = \{e_1, e_2, e_3\}$ – the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Now:

$T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3$

$T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$.

Hence

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}.$$

But if we take $\gamma' = \{e_3, e_2, e_1\}$, then $[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$.

Definition Let $T, U : V \to W$ be functions, where V, W are v.s over F, and let $a \in F$. We define:
$T+U : V \to W$ by $(T+U)(x) = T(x) + U(x)$ for all $x \in V$.
$aT : V \to W$ by $(aT)(x) = aT(x)$ for all $x \in V$.
So $T+U$ and aT are both functions from V to W.

These operations preserve linearity.

Theorem 2.7 Let V, W be v.s. over F, let $T, U : V \to W$ be linear.
a) For all $a \in F$, $aT+U$ is linear.
b) With this operations of addition and scalar multiplication, the set of all linear transformations from V to W is a v.s. over F.

Proof.
a) Let $x, y \in V$ and $c \in F$. Then

$$\underset{\text{function}}{(aT+U)}\underset{\text{vector in } V}{(cx+y)} = (aT)(cx+y) + U(cx+y) = a(T(cx+y)) + (c\,U)(x) + U(y) = a\left(cT(x) + T(y)\right) + cU(x) + U(y) =$$
$$= acT(x) + cU(x) + aT(y) + U(y) = c(aT+U)(x) + (aT+U)(y).$$

Hence the map $aT+U$ is linear.

b) Note that the zero transformation T_0 (recall $T_0(x) = 0$ for all $x \in V$) plays the role of the zero vector, and its easy to verify that all of the axioms (VS1)-(VS8) of a vector space are satisfied.

Definition For V, W v.s. over F, we denote $\mathcal{L}(V,W) = \{T : T \text{ is a lin. transf. from } V \text{ to } W\}$ – a v.s. over F.
In case $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V,W)$.

Algebraic description of the operations in $\mathcal{L}(V,W)$.

Last time we saw:
- every lin. transformation can be represented by a matrix,
- linear transformations from V to W form a vector space $\mathcal{L}(V,W)$, under pointwise addition and scalar mult.

These operations on $\mathcal{L}(V,W)$ correspond to matrix addition and scalar mult. on the representations.

Theorem 2.8
Let V, W be fin. dim. v.s. with ordered bases β and γ, respectively.
Let $T, U : V \to W$ be linear. Then:
a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. operations on matrices!
b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars $a \in F$.

Proof.
a) Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$.
There exist unique scalars a_{ij} and b_{ij} $(1 \le i \le m,\ 1 \le j \le n)$ s.t.:
$T(v_j) = \sum_{i=1}^m a_{ij} w_i$ and $U(v_j) = \sum_{i=1}^m b_{ij} w_i$ for $1 \le j \le n$.
Hence

$$(T+U)(v_j) = \sum_{i=1}^m (a_{ij} + b_{ij}) w_i.$$

Thus, for the matrix $[T+U]_\beta^\gamma$ we have

$$([T+U]_\beta^\gamma)_{ij} = a_{ij} + b_{ij} = ([T]_\beta^\gamma + [U]_\beta^\gamma)_{ij}.$$

b) Similar (Exercise.)

Example. Let $T, U : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2)$,
$U(a_1, a_2) = (a_1 - a_2,\ 2a_1,\ 3a_1 + 2a_2)$.
Let β, γ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, resp. Then

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$ – calculated in the previous example

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

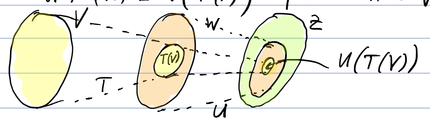
Applying definition, we have

$$(T+U)(a_1,a_2) = (a_1+3a_2, 0, 2a_1-4a_2) + (a_1-a_2, 2a_1, 3a_1+2a_2) = (2a_1+2a_2, 2a_1, 5a_1-2a_2).$$ So

$$[T+U]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix} = [T]_\beta^\gamma + [U]_\beta^\gamma$$ – as Theorem 2.8 predicts.

Composition of lin. transf.'s and matrix multiplication.

Definition. Let $T: V \to W$ and $U: W \to Z$ be two lin. transf.'s of v.s.'s. Their composition, denoted by UT, is a function from V to Z defined by $UT(x) = U(T(x))$ for all $x \in V$.



Theorem 2.9. Let V, W, Z be v.s. over F.
Let $T: V \to W$ and $U: W \to Z$ be linear.
Then $UT: V \to Z$ is linear.

Proof.
Let $x, y \in V$ and $a \in F$. Then

$$UT(ax+y) = U(T(ax+y)) \overset{(T \text{ is lin.})}{=} U(aT(x)+T(y)) \overset{(U \text{ is lin.})}{=} aU(T(x)) + U(T(y)) = a(UT)(x) + UT(y).$$

• See Problem Set 4 for more basic properties of the composition.

• Assume that V, W, Z are v.s. over F, and let $\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$, $\gamma = \{z_1, \dots, z_p\}$ be ordered bases for V, W and Z, respectively.
Let $T: V \to W$ and $U: W \to Z$ be linear.
Let $A = [U]_\beta^\gamma$ and $B = [T]_\alpha^\beta$ be their matrix representations.
We have $UT: V \to Z$ – their composition.

• Let's calculate its matrix representation $[UT]_\alpha^\gamma$.
For $1 \le j \le n$, we have

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{k=1}^m B_{kj} w_k\right) = \sum_{k=1}^m B_{kj} U(w_k) = \sum_{k=1}^m B_{kj}\left(\sum_{i=1}^p A_{ik} z_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m A_{ik} B_{kj}\right) z_i = \sum_{i=1}^p C_{ij} z_i$$

where

$$C_{ij} = \sum_{k=1}^m A_{ik} B_{kj}.$$

Hence $[UT]_\alpha^\gamma = C = (C_{ij})_{\substack{1 \le i \le p \\ 1 \le j \le n}}$.

This computation motivates the definition of matrix multiplication.

Definition. Let A be an $m \times n$ matrix, and B an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times p$ matrix such that

$$(AB_{ij}) = \sum_{k=1}^n A_{ik} B_{kj} \quad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

Example.

1) $$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 + 2\cdot 2 + 1\cdot 5 \\ 0\cdot 4 + 4\cdot 2 + (-1)\cdot 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}.$$

2) Matrix multiplication is not commutative.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$ so it is possible that $AB \ne BA$.

3) Recall the definition of the transpose of a matrix from Problem Set 2:
If $A \in M_{m\times n}(F)$, then its transpose $A^t \in M_{n \times m}(F)$ is given by $(A^t)_{ij} = A_{ji}$ for all $1 \le i \le m$, $1 \le j \le n$.
We show that

$$(AB)^t = B^t A^t.$$

Indeed, we have
$$(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} B_{ki} A_{jk} = \sum_{k=1}^{n} (B^t)_{ik} (A^t)_{kj} = (B^t A^t)_{ij}.$$

Returning to our previous calculation, we can now state it in a compact form using matrix multiplication.

Theorem 2.11.
Let V, W and Z be fin. dim. v.s. with ordered bases α, β and γ, respectively.
Let $T: V \to W$ and $U: W \to Z$ be lin. transformations. Then
$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Corollary. Let V be a fin. dim. v.s. with an ordered basis β.
Let $T, U \in \mathcal{L}(V)$. Then $[UT]_\beta = [U]_\beta [T]_\beta$.

Example. Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the lin. transf. defined by
$U(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(t)\,dt$.
Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$ be the standard ordered bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively.
We have:

$U(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$
$U(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$
$U(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$
$U(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2.$

Hence $[U]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$

Similarly, for T we have:

$T(1) = x = 0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$
$T(x) = \frac{1}{2}x^2 = 0\cdot 1 + 0\cdot x + \frac{1}{2}x^2 + 0\cdot x^3$
$T(x_2) = \frac{1}{3}x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \frac{1}{3}x^3$

Hence $[T]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$

Thus
$$[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]_\beta,$$ where $I: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is the identity transformation.

This confirms the fundamental theorem of calculus in a special case!

Definition The $n\times n$ identity matrix I_n is defined by $(I_n)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}.$
Hence $I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc.

We summarize basic properties of matrix multiplication.

Theorem 2.12. Let $A \in M_{m\times n}(F)$, $B, C \in M_{n\times p}(F)$, and $D, E \in M_{q\times m}(F)$. Then
a) $A(B+C) = AB + AC$ and $(D+E)A = DA + EA$.
b) $a(AB) = (aA)B = A(aB)$ for any scalar $a \in F$.
c) $I_m A = A = A I_n$.
d) If $\dim(V) = n$ and $I: V \to V$ is the identity transformation, then $[I]_\beta = I_n$ for any ordered basis β for V.

Proof.
See textbook.
Compare to the basic properties of the composition of lin. transformations (Theorem 2.10).

Calculating value of a lin. transf. using its matrix representation

Theorem 2.14.
Let $T: V \to W$ be linear, V, W fin. dim. v.s.'s with ordered bases β and γ, respectively.
Then, for each $u \in V$ we have

m×n matrix   n×1 matrix

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta .$$

vector in W   matrix multiplication

its coordinate vector, viewed as an m×1 matrix

Proof.
Suppose $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$ – ordered bases for V and W, respectively.
Let $x \in V$, say $x = a_1 v_1 + \dots + a_n v_n$.
That is, $[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

Let $B = [T]_\beta^\gamma$. Then

$$T(x) = a_1 T(v_1) + \dots + a_n T(v_n) = a_1 \left(\sum_{i=1}^m B_{i1} w_i \right) + \dots + a_n \left(\sum_{i=1}^m B_{in} w_i \right) = \sum_{i=1}^m \left(\sum_{j=1}^n a_j B_{ij} \right) w_i .$$

Hence

$$[T(x)]_\gamma = \begin{pmatrix} \sum_{j=1}^n a_j B_{1j} \\ \vdots \\ \sum_{j=1}^n a_j B_{mj} \end{pmatrix} = B \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$ – as wanted.

Example. Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f'(x)$.
Then

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ – calculated in a previous example
β, γ – standard ordered bases.

Let $p(x) \in P_3(\mathbb{R})$ be arbitrary, for example $p(x) = 2 - 4x + x^2 + 3x^3$.
Then $T(p(x)) = p'(x) = -4 + 2x + 9x^2$.
Hence:

$$[T(p(x))]_\gamma = [p'(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}.$$

But also

$$[T]_\beta^\gamma [p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$ – illustrating Theorem 2.14.

Associating a linear transformation to a matrix

Definition Let $A \in M_{m \times n}(F)$. We denote by L_A the mapping
$L_A : F^n \to F^m$ defined by $L_A(x) = Ax$.
regarded as column vectors.
We call L_A a left-multiplication transformation.

Example.
Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \in M_{2\times 3}(\mathbb{R})$, hence $L_A : \mathbb{R}^3 \to \mathbb{R}^2$.

If $x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ then $L_A(x) = Ax = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Theorem 2.15 (Properties of L_A)

Let $A \in M_{m\times n}(F)$. Then $L_A : F^n \to F^m$ is linear.

If $B \in M_{m\times n}(F)$ and β, γ are the standard ordered bases for F^n and F^m, resp, then:

a) $[L_A]_\beta^\gamma = A$.

b) $L_A = L_B \iff A = B$.

c) $L_{A+B} = L_A + L_B$, $L_{aA} = a\cdot L_A$ for all $a \in F$.

d) If $T : F^n \to F^m$ is lin., then there is a unique $C \in M_{m\times n}(F)$ s.t. $T = L_C$. In fact, $C = [T]_\beta^\gamma$.

e) If $E \in M_{n\times p}(F)$, then $L_{AE} = L_A L_E$.

f) If $m = n$, then $L_{I_n} = I_{F^n}$.

Proof. Linearity of L_A is clear by Theorem 2.12.

a) The j^{th} column of $[L_A]_\beta^\gamma$ is $L_A(e_j) = Ae_j$, which is also the j^{th} column of A.
So $[L_A]_\beta^\gamma = A$.

b) "$\Leftarrow$": clear
"$\Rightarrow$": If $L_A = L_B$, then by (a), $A = [L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$.

d) Let $T : F^n \to F^m$ be lin, let $C = [T]_\beta^\gamma$.
By Theorem 2.14,
$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$, or $T(x) = Cx = L_C(x)$ for all $x \in F^n$.
So $T = L_C$. The uniqueness of C follows from (b).

e) $(AE)e_j$ = the j^{th} column of $AE = A(Ee_j)$ — both equalities are easy to see by writing out the products.

Thus $L_{AE}(e_j) = (AE)e_j = A(Ee_j) = L_A(Ee_j) = L_A(L_E(e_j))$.

Hence $L_{AE} = L_A L_E$ (by the corollary to Theorem 2.6, if to linear transf.'s agree on a basis, then they are equal).

(c), (f) — Exercise.

Theorem 2.16. (Matrix multiplication is associative)

Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$, $C \in M_{p\times r}(F)$. Then
$A(BC) = (AB)C$.

Proof.

We have (using Theorem 2.15(e) and associativity of the composition of functions)
$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB} L_C = L_{(AB)C}$.

By Theorem 2.15 (b), $A(BC) = (AB)C$.

Invertibility

Definition. Let V and W be v.s. and $T : V \to W$ linear.
A function $U : W \to V$ is an inverse of T if $TU = I_W$ and $UT = I_V$.
If T has an inverse, then T is invertible.
If T is invertible, then the inverse of T is unique and is denoted by T^{-1}.

Basic facts about invertible functions.

1) Let T and U be invertible. Then the following holds:
a) $(TU)^{-1} = U^{-1}T^{-1}$
b) $(T^{-1})^{-1} = T$; in particular, T^{-1} is invertible.

2) T is invertible $\iff$ T is a bijection.

Proof. 2) "$\Rightarrow$" For any $y \in W$, $TT^{-1}(y) = I_W(y) = y$. Hence $y = T(\underbrace{T^{-1}(y)}_{\in V})$, so T is surjective.
Assume $T(x_1) = T(x_2)$, then $T^{-1}(T(x_1)) = T^{-1}(T(x_2))$, hence $x_1 = x_2$ — so T is injective.

Theorem 2.17. Let V, W be v.s., let $T: V \to W$ be lin. and invertible.
Then $T^{-1}: W \to V$ is also linear.

Proof.
Let $y_1, y_2 \in W$ and $c \in F$. Since T is both surjective and injective, there exist <u>unique</u> vectors $x_1, x_2 \in V$ s.t. $T(x_1) = y_1$ and $T(x_2) = y_2$.
Thus $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$. And so
$T^{-1}(cy_1 + y_2) = T^{-1}(cT(x_1) + T(x_2)) = T^{-1}(T(cx_1 + x_2)) = I_V(cx_1 + x_2) = cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2)$.

Example. Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be the lin. transf. defined by $T(a + bx) = (a, a+b)$.
Then $T^{-1}: \mathbb{R}^2 \to P_1(\mathbb{R})$ is defined by $T^{-1}(c, d) = c + (d-c)x$ — also linear, as Theorem 2.17 predicts.

- Recall the analogy between linear transformations and matrices.

Definition. Let $A \in M_{n\times n}(F)$. Then A is invertible if there exists $B \in M_{n \times n}(F)$ s.t. $AB = BA = I$.

Note. If A is invertible, then the matrix B such that $AB = BA = I$ is unique, called the inverse of A and denoted A^{-1}.
(If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$).

Example. The inverse of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Indeed, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Lemma. Let $T: V \to W$ be lin. and invertible, and $\dim(V) < \infty$. Then $\dim(V) = \dim(W)$.

Proof. Let $\beta = \{x_1, \dots, x_n\}$ be a basis for V.
By Theorem 2.2, $\mathrm{Span}(T(\beta)) = R(T) = W$.
Next, T is a bijection, so:
$\dim(N(T)) = 0$ (as $N(T) = \{0\}$ as T is injective).
$\dim(R(T)) = \dim(W)$ (as $R(T) = W$).
Hence, by the dimension theorem, $\dim(V) = \dim(N(T)) + \dim(R(T)) = \dim(W)$.

Theorem 2.18 Let V, W be fin. dim. v.s. with ordered bases β and γ, resp.
Let $T: V \to W$ be lin.
Then T is invertible $\iff$ $[T]_\beta^\gamma$ is invertible.
Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Proof.
"$\Rightarrow$" Suppose T is invertible.
By the Lemma, $\dim(V) = \dim(W) = n$. So $[T]_\beta^\gamma \in M_{n\times n}(F)$.
By definition, $T^{-1}: W \to V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$. Thus
$I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$.
Similarly,
$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$.
So $[T]_\beta^\gamma$ is invertible and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

"$\Leftarrow$" Suppose $A = [T]_\beta^\gamma$ is invertible. Then there exists $B \in M_{n\times n}(F)$ s.t. $AB = BA = I_n$.
By Theorem 2.6, there exists $U \in \mathcal{L}(W, V)$ s.t.
$U(w_j) = \sum_{i=1}^n B_{ij} v_i$ for $j = 1, \dots, n$,
where $\gamma = \{w_1, \dots, w_n\}$, $\beta = \{v_1, \dots, v_n\}$.
It follows that $[U]_\gamma^\beta = B$.
To show that $U = T^{-1}$, notice that
$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$ — by Theorem 2.11.
So $UT = I_V$, and similarly, $TU = I_W$.

Example. Let β and γ be the standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$, resp.
For T given by $T(a+bx) = (a, a+b)$ from the previous example, we have
$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $[T^{-1}]_\gamma^\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. We have already checked that each of these matrices is the inverse of the other.

Corollary. Let $A \in M_{n\times n}(F)$. Then A is invertible $\iff$ L_A is invertible. Moreover, $(L_A)^{-1} = L_{A^{-1}}$.

Isomorphisms.

Sometimes two vector spaces may consist of objects of very different nature, but behave identically from the algebraic point of view. We describe a precise way of "identifying" vector spaces with each other.

Definition. Let V, W be v.s. We say that V is *isomorphic* to W if there exists a lin. transf. $T: V \to W$ that is invertible.
Such a lin. transf. is called an *isomorphism* from V onto W.

Note. 1) V is isomorphic to V (using I_V).
2) V is isomorphic to W $\iff$ W is isomorphic to V.
3) If V is isomorphic to W and W is isomorphic to Z, then V is isomorphic to Z. Exersice.
Thus isomorphism is an *equivalence relation* on vector spaces.

Example. Let $T: F^2 \to P_1(F)$ be given by $T(a_1, a_2) = a_1 + a_2 x$.
Then T is an isomorphism, so F^2 is isomorphic to $P_1(F)$.

Theorem 2.19. Let V, W be fin. dim. v.s. over F.
Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Proof. "$\Rightarrow$" Let $T: V \to W$ be an isomorphism from V to W.
By the lemma above, $\dim(V) = \dim(W)$.

"$\Leftarrow$" Suppose $\dim(V) = \dim(W)$, and let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_n\}$ be bases for V and W, resp.
By Theorem 2.6, there exists $T: V \to W$ s.t. T is lin. and $T(v_i) = w_i$ for $i = 1, \dots, n$.
By Theorem 2.2,
$R(T) = \mathrm{Span}(T(\beta)) = \mathrm{Span}(\gamma) = W$, so T is surjective.
By Theorem 2.5, T is also injective.
Hence T is an isomorphism.

Corollary. Let V be a v.s. over F.
Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Up to this point, we have associated linear transformations with their matrix representations, and we have seen many analogies between the operations on $\mathcal{L}(V,W)$ and $M_{m\times n}(F)$.
Now we can show that these two spaces may be identified.

Theorem 2.20.
Let V, W be v.s. over F, $\dim(V) = n$, $\dim(W) = m$.
Let β, γ be ordered bases for V and W, respectively.
Then the function $\Phi: \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by
$\Phi(T) = [T]_\beta^\gamma$ for all $T \in \mathcal{L}(V,W)$
is an isomorphism.

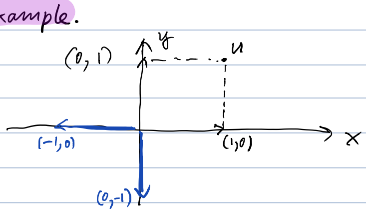
Proof.
By Theorem 2.8, Φ is linear. So remains to show Φ is a bijection.
That is, we need to show that for every $A \in M_{m\times n}(F)$, there is a unique lin. transf. $T: V \to W$ s.t. $\Phi(T) = A$.
Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$, and let $A \in M_{m\times n}(F)$ be given.
By Theorem 2.6, there exists a unique lin. transf. $T: V \to W$ s.t.
$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$ for $1 \le j \le n$.
But this means that $[T]_\beta^\gamma = A$, or $\Phi(T) = A$. Thus Φ is an isomorphism.

Corollary. If $\dim(V) = n$, $\dim(W) = m$, then $\dim(\mathcal{L}(V, W)) = mn$.
(by the previous theorem, as $\dim(M_{m\times n}(F)) = mn$).

Change of coordinate matrix

We have seen that once we fix an ordered basis β of a v.s. V to every vector $u \in V$ we can assign its coordinates $[u]_\beta$. And similarly, for $T: V \to V$, we assign its matrix rep. $[T]_\beta$.
Hower, these coordinates depend on β! And can be different for another choice of an ordered basis.

Example.



$V = \mathbb{R}^2$
$\beta = \{(1,0), (0,1)\}$ – ordered basis
$[u]_\beta = \begin{pmatrix}1\\1\end{pmatrix}$
$\gamma = \{(-1,0), (0,-1)\}$ – another ordered basis for V.
$[u]_\gamma = \begin{pmatrix}-1\\-1\end{pmatrix}$.

We would like a method to calculate $[u]_\gamma$ from $[u]_\beta$, for an arbitrary choice of β and γ.

Definition. Let β and β' be two ordered bases for a fin. dim. v.s. V.
We define the change of coordinate matrix (or "change of basis matrix") to be $Q = [I_V]_{\beta'}^{\beta}$.

Theorem 2.22.
1) Q is invertible. (and $Q^{-1} = [I_V]_\beta^{\beta'}$).
2) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Proof.
1) As I_V is invertible, Q is also invertible by Thm 2.18
2) For any $v \in V$,
$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta} [v]_{\beta'} = Q[v]_{\beta'}$, by Theorem 2.14.

So, multiplying by Q changes the β'-coordinates of a vector into its β-coordinates.
And multiplying by Q^{-1} changes β-coordinates into β'-coordinates.

Example.
In the example above, $[I_V]_\gamma^\beta = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$.
And $[u]_\gamma = \begin{pmatrix}-1\\-1\end{pmatrix}$.
Hence $[u]_\beta = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

Definition A lin. transf. $T: V \to V$ from a v.s. V to itself is called a linear operator on V.
• Now we determine how to calculate $[T]_\beta$ from $[T]_{\beta'}$, for β, β' two ordered bases for V.

Theorem 2.23. Let T be a lin. operator on a fin. dim. v.s. V.
Let β, β' be ordered bases for V.
Let $Q = [I_V]_{\beta'}^{\beta}$ be the change of coordinate matrix, changing β'-coord's into β-coords.
Then
$[T]_{\beta'} = Q^{-1} [T]_\beta Q$.

Proof.
Recall that $T = I_V T = T I_V$.
$Q [T]_{\beta'} = [I_V]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} = [I_V T]_{\beta'}^{\beta} = [T I_V]_{\beta'}^{\beta} = [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta} = [T]_\beta Q$. (by Theorem 2.11).
Therefore
$[T]_{\beta'} = Q^{-1} [T]_\beta Q$.

Example.
Consider the lin. operator T on $\mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$.
Let $\beta = \{(1,0),(0,1)\}$ and $\beta' = \{(-1,0),(0,-1)\}$ be ordered bases.
By the previous example:
$Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Also $Q^{-1} = [I_V]_{\beta}^{\beta'} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Also $[T]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Hence $[T]_{\beta'} = Q^{-1} [T]_\beta Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.