Determinants

Definition

Let $A \in M_{n\times n}(F)$.

1) For any $1 \le i, j \le n$ we define the cofactor matrix of the entry of A in row i and column j to be the matrix $\tilde{A}_{ij} \in M_{(n-1)\times(n-1)}(F)$ obtained from A by deleting row i and column j.

2) The determinant of A, denoted $\det(A)$, is a scalar in F defined recursively as follows:
- if $n=1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$.
- for $n \ge 2$, we define

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}) \quad \text{for any } 1 \le i \le n$$

(this formula gives the same value for any i! See Theorem 4.4).

3) Equivalently, we have

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij}) \quad \text{for any } 1 \le j \le n.$$

Example. Let's consider the case $n=2$.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{2\times 2}(F)$ be given.

According to the definition, we can evaluate its determinant along any row i.

Let's take $i=1$.

Then the cofactor matrices are $\tilde{A}_{1,1} = (A_{22})$ and $\tilde{A}_{1,2} = (A_{21})$.

So $\det(\tilde{A}_{1,1}) = A_{22}$, $\det(\tilde{A}_{1,2}) = A_{21}$ and

$$\det(A) = \sum_{j=1}^{2} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1,j}) = A_{11} \cdot A_{22} - A_{12} A_{21}.$$

– the familiar formula.

Example.

Let $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

Again, let's calculate $\det(A)$ using cofactors along the 1^{st} row. We obtain:

$$\det(A) = (-1)^{1+1} A_{11} \det(\tilde{A}_{1,1}) + (-1)^{1+2} A_{12} \cdot \det(\tilde{A}_{1,2}) + (-1)^{1+3} A_{13} \det(\tilde{A}_{1,3}) =$$

$$= (-1)^2 \cdot 1 \cdot \det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3 \cdot 3 \cdot \det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + (-1)^4 \cdot (-3) \cdot \det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} =$$

$$= 1 \cdot (-5 \cdot (-6) - 2 \cdot 4) - 3(-3 \cdot (-6) - 2 \cdot (-4)) - 3(-3 \cdot 4 - (-5) \cdot (-4)) =$$

$$= 1 \cdot 22 - 3 \cdot 26 - 3 \cdot (-32) = 40.$$

Properties of the determinant (See Sections 4.2-4.4 in the textbook for the proofs)

Let $A \in M_{n\times n}(F)$. If B is a matrix obtained from A by

1) switching two rows (or two columns), then
$\det(B) = -\det(A)$.

2) multiplying a row (or a column) of A by a scalar $c \in F$, then
$\det(B) = c \cdot \det(A)$.

3) adding a multiple of row i to row j (or a multiple of column i to column j), then
$\det(B) = \det(A)$.

These properties are helpful for computing determinants.
We also have the following properties:

4) If $B \in M_{n\times n}(F)$, then
$$\det(AB) = \det(A)\cdot\det(B),$$

5) A is invertible if and only if $\det(A) \neq 0$. Furthermore,
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

6) If $I_n \in M_{n\times n}(F)$ is the identity matrix, then
$$\det(I_n) = 1.$$

7) $\det(A) = \det(A^t)$.

The operations on the rows of a matrix described in 1), 2) and 3) above are called elementary row operations.

Fact. Using these operations, we can transform any square matrix into an upper triangular matrix. That is, a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ 0 & & \ddots & \\ & & & A_{nn} \end{pmatrix} \quad - \text{ all entries below the diagonal are } 0.$$

Fact. If $A \in M_{n\times n}(F)$ is upper triangular, then $\det(A) = A_{11}\cdot A_{22}\cdot \ldots \cdot A_{nn}$.

These two facts simplify calculating the determinants.

Example.
Let $B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$.

Applying elementary rows operations, we have

$$B \xrightarrow{(1)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}.$$

exchanging rows 1 and 2   adding $2\times$(row 1) to row 3   adding $10\times$(row 2) to row 3

As (3) doesn't change the determinant, we have
$\det\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} = -2\cdot 1\cdot(24) = -48$, and as (1) only changes the sign of det, we have $\det(B) = 48$.

Eigenvalues and eigenvectors.

Definition. 1) A lin. operator T on a fin. dim. v.s. V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. That is,

$$[T]_\beta = \begin{pmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & \ddots \\ & & & A_{nn} \end{pmatrix} \quad \text{for some } A_{11}, \ldots, A_{nn} \in F.$$

2) A matrix $A \in M_{n\times n}(F)$ is diagonalizable if A is similar to a diagonal matrix.
Recall: two matrices $A, B \in M_{n\times n}(F)$ are similar if there is an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^{-1}AQ$.

Theorem. Let $T: V \to V$ be a lin. operator, $\dim(V) < \infty$ and β, γ ordered bases for V.
Then $\det([T]_\beta) = \det([T]_\gamma)$.

Proof.
There exists an invertible matrix Q s.t. $[T]_\gamma = Q^{-1}[T]_\beta Q$ (namely, the change of coordinates matrix $[I_V]_\gamma^\beta$ converting γ-coordinates to β-coordinates).
Then, using the basic properties of det, we have:
$\det([T]_\gamma) = \det(Q^{-1}[T]_\beta Q) = \det(Q^{-1})\cdot\det([T]_\beta)\cdot\det(Q) = (\det Q)^{-1}\cdot\det Q\cdot\det([T]_\beta) =$
$= \det([T]_\beta)$.

Definition. For a lin. operator T, we define its determinant, $\det T$, as follows: choose any ordered basis β for V and take $\det T = \det([T]_\beta)$.
(by the previous theorem, the choice of β doesn't matter).

Proposition
a) T is bijective $\iff \det T \neq 0$.
b) T is bijective $\implies \det(T^{-1}) = (\det T)^{-1}$.
c) If $U: V\to V$ is another lin. operator on V, then $\det(TU) = \det T\cdot\det U$.

Proof
Exercise, follows from the analogous properties of the matrix determinant.

Theorem. Let $T: V\to V$ be a lin. operator, $\dim(V)<\infty$, β an ordered basis for V. Then:
T is diagonalizable $\iff [T]_\beta$ is a diagonalizable matrix.

Proof.
Let $\beta = \{v_1, \dots, v_n\}$.
$\Rightarrow$ Assume that T is diagonalizable. This means that there is an ordered basis γ for V such that $D = [T]_\gamma$ is a diagonal matrix. Let $[I_V]_\gamma^\beta$ be the change of coordinates matrix.
Then $[T]_\gamma = Q^{-1}[T]_\beta Q$, so $[T]_\gamma$ and $[T]_\beta$ are similar, so $[T]_\beta$ is diagonalizable.
$\Leftarrow$ Exercise.

Corollary. $A\in M_{n\times n}(F)$ is diagonalizable $\iff L_A$ is diagonalizable.

Problem. When is A/T diagonalizable?

Theorem. T is diagonalizable $\iff$ there is an ordered basis $\beta = \{v_1,\dots,v_n\}$ for V and scalars $\lambda_1,\dots,\lambda_n\in F$ such that
$T(v_j) = \lambda_j v_j$ for $1\le j\le n$.

Proof.
If $D = [T]_\beta$ is a diagonal matrix, then for each vector $v_j\in\beta$ we have
$T(v_j) = \sum_{i=1}^n D_{ij}v_i = D_{jj}v_j = \lambda_j v_j$, where $\lambda_j = D_{jj}$.
Conversely, if β is an ord. basis for V s.t. $T(v_j) = \lambda_j v_j$, then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

This argument motivates the following definition.

Definition.
1) A non-zero vector $v\in V$ is an eigenvector of T if $T(v) = \lambda v$ for some $\lambda\in F$.
We call λ the eigenvalue of T corresponding to the eigenvector v.
2) Let $A\in M_{n\times n}(F)$. A non-zero $v\in F^n$ is an eigenvector of A if
$Av = \lambda v$ for some $\lambda\in F$.
And λ is the eigenvalue of A corresponding to the eigenvector v.

3) The elements in a basis β as in the last theorem are eigenvectors, and the λ_i's are the respective eigenvalues.

Theorem 5.2. A scalar $\lambda \in F$ is an eigenvalue of $T \iff \det(T - \lambda I_V) = 0$

Proof. We have

$\lambda \in F$ is an eigenvalue of $T \iff T(v) = \lambda v$ for some $v \neq 0$ in $V \iff \overbrace{(T - \lambda I_V)}^{\text{lin. operator on } V}(v) = 0$ for some $v \neq 0$ in $V \iff N(T - \lambda I_V) \neq \{0\} \overset{\text{Thm 2.4, 2.5}}{\iff} T - \lambda I_V$ is not bijective $\underset{\text{properties of det.}}{\iff} \det(T - \lambda I_V) = 0$.

Corollary. Let $A \in M_{n \times n}(F)$. Then $\lambda \in F$ is an eigenvalue of $A \iff \det(A - \lambda I_n) = 0$.

Example. Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

Then $\det(A - \lambda I_2) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = (\lambda - 3)(\lambda + 1)$.

Hence by the corollary, the eigenvalues of A are the solutions to $(\lambda-3)(\lambda+1) = 0$ — which are $3, -1$.

Definition 1) The polynomial $f(t) = \det(A - t I_n)$ in the variable t is called the characteristic polynomial of A.

2) Given a lin. operator $T: V \to V$, $\dim(V) < \infty$, and β an ordered basis for V, we define the characteristic polynomial of T to be the char. polynomial of $A = [T]_\beta$:

$f(t) = \det(A - tI)$.

Note. Similar matrices have the same char. polynomial, so is well defined.

Properties of char. polynomial.

Let $A \in M_{n\times n}(F)$ be given, and let $f(t)$ be its char. polynomial.

1) $f(t)$ is a polynomial of degree n with leading coefficient $(-1)^n$:

$f(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \dots + c_0$ for some $c_0, \dots, c_{n-1} \in F$.

2) A scalar $\lambda \in F$ is an eigenvalue of $A \iff f(\lambda) = 0$.

3) A has at most n distinct eigenvalues (as $f(t)$ has at most n roots).

4) If $\lambda \in F$ is an eigenvalue of A, then a vector $x \in F^n$ is an eigenvector of A corresp. to $\lambda \iff x \neq 0$ and $x \in N(L_A - \lambda I_{F^n})$.

Example. Let's consider $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ again, and let's find it

1) The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

2) Let $B_1 = A - \lambda_1 I_2 = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$.

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ is an eigenvector of A corresponding to λ_1 — by (4) above.

$\iff x \neq 0$ and $x \in N(L_{B_1}) \iff x \neq 0$ and $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\iff$

$\iff \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The set of all solutions to this system of lin. equations is

$\left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\}$.

Hence $x \in \mathbb{R}^2$ is an eigenvector corresp. to $\lambda_1 = 3 \iff x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for some $t \neq 0$.

3) Let $B_2 = A - \lambda_1 I_2 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$. Hence:

$x \in \mathbb{R}^2$ is an e. vec. of A corresp. to $\lambda_2 \iff x \neq 0$ and $x \in N(L_{B_2}) \iff B_2 \cdot x = 0 \iff \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\iff \begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$.

Hence $N(L_{B_2}) = \{t\binom{1}{-2} : t \in \mathbb{R}\}$. Thus x is an e.vec. corresp. to $\lambda_2 = -1$ $\iff$ $x = t\binom{1}{-2}$ for some $t \neq 0$.

Notice that $\{\binom{1}{2}, \binom{1}{-2}\}$ is a basis for $\mathbb{R}^2$ consisting of e.vectors of A. Thus L_A, and hence A, is diagonalizable.

Determining eigenvectors and eigenvalues of a lin. operator

Let V be a v.s., $\dim(V) = n$. Let β be an ordered basis for V.
Let $T \in \mathcal{L}(V)$ be a lin. operator on V.

Summarizing the results of the previous section, we describe how to determine the e.val's and the e.vec's of T.

1) Determine the matrix representation $[T]_\beta$ of T.
2) Determine the e.val's of T.
   $\lambda \in F$ is an e.val of T $\iff$ λ is a root of the char. polynomial of T.
   That is, we need to find the solutions $x \in F$ of $\det([T]_\beta - x I_n) = 0$.
   There are at most n distinct solutions $\lambda_1, \dots, \lambda_n$.
3) Now for each e.val. λ of T, we can determine the corresponding e.vec's. We have:
   $T(v) = \lambda v \iff (T - \lambda I_V)(v) = 0 \iff [T - \lambda I_V]_\beta [v]_\beta = 0$.
   Therefore, eigenvectors corresponding to λ are the solutions of this system of linear equations. (more precisely, solving this system we find the β-coordinates $[v]_\beta$, which then determines v).