MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 1

DUE FRIDAY, APRIL 12

Problem 1. Let f be one of the following functions on a vector space V, determine (with demonstration) if it is a linear functional.

(1) $V = P(\mathbb{R})$, $f(p(x)) = 2p'(0) + p''(1)$, where $'$ denotes differentiation.
(2) $V = \mathbb{R}^2$, $f(x, y) = (2x, 4y)$.
(3) $V = M_{2\times 2}(F)$, $f(A) = \operatorname{tr}(A)$.
(4) $V = P(\mathbb{R})$, $f(p(x)) = \int_0^1 p(t)\,dt$.
(5) $V = \mathbb{R}^3$, $f(x, y, z) = x^2 + y^2 + z^2$.

Problem 2. For each of the following vector spaces V and ordered bases β, find explicit formulas for vectors of the dual basis β^* for V^*.

(1) $V = \mathbb{R}^3$, $\beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$.
(2) $V = P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$.

Problem 3. Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows:

- $f_1(x, y, z) = x - 2y$,
- $f_2(x, y, z) = x + y + z$,
- $f_3(x, y, z) = y - 3z$.

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

Problem 4. Define $f \in (\mathbb{R}^2)^*$ by $f(x, y) = 2x + y$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (3x + 2y, x)$.

(1) Compute $T^t(f)$.
(2) Compute $[T^t]_{\beta^*}$, where β is the standard ordered basis for $\mathbb{R}^2$ and $\beta^* = \{f_1, f_2\}$ is the dual basis, by finding scalars a, b, c and d such that $T^t(f_1) = af_1 + cf_2$ and $T^t(f_2) = bf_1 + df_2$.
(3) Compute $[T]_\beta$ and $\left([T]_\beta\right)^t$, and compare your results with (2).

Problem 5. Prove that a function $T : F^n \to F^m$ is linear if and only if there exist $f_1, f_2, \ldots, f_m \in (F^n)^*$ such that $T(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for all $x \in F^n$.

(Hint: if T is linear, define $f_i(x) = (g_i T)(x)$ for $x \in F^n$; that is, $f_i = T^t(g_i)$ for $1 \le i \le m$, where $\{g_1, g_2, \ldots, g_m\}$ is the dual basis of the standard ordered basis for F^m.)

Problem 6. Let V be a finite-dimensional vector space over F. For every subset S of V, define the *annihilator* S^0 of S as

$$S^0 = \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}.$$

(1) Prove that S^0 is a subspace of V^*.
(2) If W is a subspace of V and $x \notin W$, prove that there exists $f \in W^0$ such that $f(x) \neq 0$.

(3) Prove that $\left(S^0\right)^0 = \mathrm{Span}\left(\psi\left(S\right)\right)$, where ψ is defined as in Theorem 2.26.
(4) For subspaces W_1 and W_2 of V, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.
(5) For subspaces W_1 and W_2, prove that $\left(W_1 + W_2\right)^0 = W_1^0 \cap W_2^0$.

Problem 7. Prove that if W is a subspace of V, then $\dim\left(W\right) + \dim\left(W^0\right) = \dim\left(V\right)$.

(Hint: extend an ordered basis $\{x_1, \ldots, x_k\}$ of W to an ordered basis $\beta = \{x_1, \ldots, x_k, \ldots, x_n\}$ of V. Let $\beta^* = \{f_1, \ldots, f_n\}$. Prove that $\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is a basis for W^0.)

Problem 8. Suppose that W is a finite-dimensional vector space and $T : V \to W$ is a linear transformation. Prove that $N\left(T^t\right) = \left(R\left(T\right)\right)^0$.