MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 5

DUE FRIDAY, MAY 10

Problem 1. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither.

(1) $V = \mathbb{R}^2$ and T is defined by $T(a, b) = (2a - 2b, -2a + 5b)$.
(2) $V = \mathbb{C}^2$ and T is defined by $T(a, b) = (2a + ib, a + 2b)$.
(3) $V = P_2(\mathbb{R})$ and T is defined by $T(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt.$$

(4) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$.

Problem 2. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

Problem 3. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*).$$

(1) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.
(2) Suppose also that $T = U_1 + iU_2$, where $U_1$and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.
(3) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

Problem 4. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(1) If T is self-adjoint, then T_W is self-adjoint.
(2) $W^\perp$ is T^*-invariant.
(3) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(4) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Problem 5. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant.

Problem 6. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

Problem 7. Assume that T is a linear operator on a complex finite-dimensional inner product space V with an adjoint T^*. Prove the following results.

(1) If T is self-adjoint, then $\langle T(x), x \rangle$ is real for all $x \in V$.
(2) If T satisfies $\langle T(x), x \rangle = 0$ for all $x \in V$, then $T = T_0$.
(Hint: replace x by $x + y$ and then by $x + iy$, and expand the resulting inner products.)
(3) If $\langle T(x), x \rangle$ is real for all $x \in V$, then $T = T^*$.

Problem 8. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Conclude that T is self-adjoint.