MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 6

DUE FRIDAY, MAY 17

Problem 1. Prove Corollary 2 to theorem 6.18.

That is, prove that if T is a linear operator on a finite-dimensional complex inner product space V, then: V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 **if and only if** T is unitary.

Problem 2. Recall: a matrix $A \in M_{n\times n}(F)$ is *unitarily (orthogonally) equivalent* to $B \in M_{n\times n}(F)$ if there exists a unitary (orthogonal) matrix $P \in M_{n\times n}(F)$ such that $A = P^*BP$. Prove that this is an equivalence relation on $M_{n\times n}(F)$.

Problem 3. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \ldots + \lambda_k T_k$ of T, $\lambda_i \in \mathbb{C}$, to prove the following.

(1) If g is a polynomial over $\mathbb{C}$, then $g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i$.
(2) If $T^n = T_0$ for some n, then $T = T_0$.
(3) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.
(4) There exists a normal operator U on V such that $U^2 = T$.
(5) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.
(6) T is a projection if and only if every eigenvalue of T is 1 or 0.
(7) $T = -T^*$ if and only if every λ_i is an imaginary number.

Problem 4. Show that if T is a normal operator on a complex finite-dimensional inner product space and U is a linear operator that commutes with T, then U also commutes with T^*.

Problem 5. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then it is also an orthogonal projection.

Problem 6. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that:

(1) $U(W) = W$;
(2) $W^\perp$ is U-invariant.

Problem 7. Prove part (c) of the spectral theorem.

Problem 8. Let V be a finite-dimensional real inner product space. Prove that rotations, reflections and compositions of rotations and reflections are orthogonal operators.