MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 7

DUE FRIDAY, MAY 24

Problem 1. Let T be an orthogonal (unitary) operator on a finite-dimensional real (respectively, complex) inner product space V. If W is a T-invariant subspace of V, prove the following.

(1) T_W is an orthogonal (respectively, unitary) operator on W.
(2) $W^{\perp}$ is a T-invariant subspace of V.
(Hint: use the fact that T_W is one-to-one and onto to conclude that for any $y \in W$, $T^{*}(y) = T^{-1}(y) \in W$.)
(3) $T_{W^{\perp}}$ is an orthogonal (respectively, unitary) operator.

Problem 2. Let T be a linear operator on a finite-dimensional vector space V, where $V = W_1 \oplus \ldots \oplus W_k$ is a direct sum of T-invariant subspaces W_i of V. Prove that $\det(T) = \det(T_{W_1}) \cdot \ldots \cdot \det(T_{W_k})$.

Problem 3. Complete the proof of the Corollary to Theorem 6.47.

Problem 4. For any real number $\phi \in \mathbb{R}$, let

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

(1) Prove that L_A is a reflection.
(2) Find the subspace of $\mathbb{R}^2$ about which L_A reflects.

Problem 5. For any real number ϕ, define $T_\phi = L_A$ for

$$A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

(1) Prove that any rotation on $\mathbb{R}^2$ is of the form T_ϕ for some ϕ.
(2) Prove that $T_\phi T_\psi = T_{(\phi+\psi)}$ for any $\phi, \psi \in \mathbb{R}$.
(3) Deduce that any two rotations on $\mathbb{R}^2$ commute.

Problem 6.

(1) Prove that the composite of any two rotations on $\mathbb{R}^3$ is a rotation on $\mathbb{R}^3$.
(2) Prove that the composite of any two reflections on $\mathbb{R}^3$ is a rotation on $\mathbb{R}^3$.
(3) Give an example of an orthogonal operator that is neither a reflection nor a rotation.

Problem 7. Prove that no orthogonal operator can be both a rotation and a reflection.

Problem 8. Let V be a finite-dimensional real inner product space. Define $T : V \to V$ by $T(x) = -x$. Prove that T is a product of rotations if and only if $\dim(V)$ is even.