MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 8

DUE FRIDAY, MAY 31

Problem 1. Let V be a real inner product space of dimension 2. For any $x, y \in V$ such that $x \neq y$ and $\|x\| = \|y\| = 1$, show that there exiss a unique rotation T on V such that $T(x) = y$.

Problem 2. Let T be a linear operator on an n-dimensional real inner product space V. Suppose that T is not the identity. Prove the following.

(1) If n is odd, then T can be expressed as the composite of at most one reflection and at most $\frac{1}{2}(n-1)$ rotations.
(2) If n is even, then T can be expressed as the composite of at most $\frac{1}{2}n$ rotations or as the composite of one reflection and at most $\frac{1}{2}(n-2)$ rotations.

Problem 3. Let T be a linear operator on V and $\beta = \{v_1, \ldots, v_n\}$ is a canonical Jordan basis, i.e. $[T]_\beta$ is a canonical Jordan form. Show that for each $i = 1, \ldots, n$ there is some $p \in \mathbb{N}$ such that $(T - \lambda I)^p (v_i) = 0$, where λ is the diagonal entry of the matrix $[T]_\beta$ on the i^{th} column.

Problem 4. Let $T : V \to W$ be a linear transformation. Prove the following.

(1) $N(T) = N(-T)$.
(2) $N\left(T^k\right) = N\left((-T)^k\right)$.
(3) If $V = W$ and λ is an eigenvalue of T, then for any positive integer k

$$N\left((T - \lambda I_V)^k\right) = N\left((\lambda I_V - T)^k\right).$$

Problem 5. Let U be a linear operator on a finite-dimensional vector space V. Prove the following.

(1) $N(U) \subseteq N\left(U^2\right) \subseteq \ldots \subseteq N\left(U^k\right) \subseteq N\left(U^{k+1}\right) \subseteq \ldots$.
(2) If $\operatorname{rank}(U^m) = \operatorname{rank}\left(U^{m+1}\right)$ for some positive integer m, then $\operatorname{rank}(U^m) = \operatorname{rank}\left(U^k\right)$ for any positive integer $k \geq m$.
(3) If $\operatorname{rank}(U^m) = \operatorname{rank}\left(U^{m+1}\right)$ for some positive integer m, then $N(U^m) = N\left(U^k\right)$ for some positive integer $k \geq m$.
(4) Let T be a linear operator on V, and let λ be an eigenvalue of T. Prove that if $\operatorname{rank}\left((T - \lambda I)^m\right) = \operatorname{rank}\left((T - \lambda I)^{m+1}\right)$ for some integer m, then $K_\lambda = N\left((T - \lambda I)^m\right)$.
(5) Let $T \in \mathcal{L}(V)$ have a characteristic polynomial that splits, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then T is diagonalizable if and only if $\operatorname{rank}(T - \lambda I) = \operatorname{rank}\left((T - \lambda I)^2\right)$ for $1 \leq i \leq k$.

Problem 6. Use Problem 5(5) to obtain a new proof of the following result from previous homework: if $T \in \mathcal{L}(V)$ is diagonalizable, $\dim(V) < \infty$ and W is a T-invariant subspace of V, then T_W is diagonalizable.

Problem 7. Use Theorem 7.4 to prove that the vectors $v_1, \ldots, v_k$ in the statement of Theorem 7.3 are unique.